Math3250 Combinatorics Reading HW 16

Instruction.

- Submit your homework by email (subject: Math3250 Combinatorics Reading HW 16). Either hand-write (use a scanner app to convert to PDF) or type your work.
- Ref: textbook *Combinatorics and Graph Theory* by Harris, Hirst, and Mossinghoff (HHM) Sec 1.3

1. Sec 1.3.4 Counting trees

Either watch the lecture video of Sec 1.3.4 Counting trees (25 minutes) or read Sec 1.3.4 from lecture notes for Sec 1.3 video on trees. Write down what you did. If you watched the video, please specify (Kaltura/YouTube) and type of device. The video is at original speed — you can play the video at faster speed if you are not taking notes.

2. Exercises

a.) Determine the Prüfer sequence for each of the two trees below.

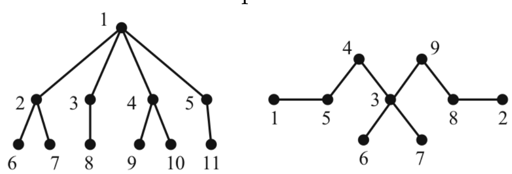

FIGURE 1.50. Two labeled trees.

b.) Draw and label a tree whose Prüfer sequence is $5, 4, 3, 5, 4, 3, 5, 4, 3$.

c.) Which trees have constant Prüfer sequence? What trees have Prüfer sequences with distinct terms?

d.) (Optional) Read the following.
Recall from Linear Algebra that the following matrix operation does not change the determinant of a matrix: replacing row i with the sum of row i and row j.

We will use the Matrix Tree Theorem to prove that the number of spanning trees of the labeled tree K_5 is 5^3. First, we compute $D - A$.

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad D - A = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

Let $M = D - A$. The $1,1$ cofactor of M is $(-1)^{1+1} det(M(1 \mid 1))$.

$$M(1 \mid 1) = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

To simplify the computation of $det(M(1 \mid 1))$, we will perform matrix operation which does not alter the value of the determinant. First, replace the first row with the sum of all rows. We get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

Next, replace each row i (except for the first row) with the sum of row i and the first row. We get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

The determinant of the above matrix is $5 \cdot 5 \cdot 5 = 5^3$.

e.) Use the Matrix Tree Theorem to prove Cayley's Theorem. (Do the above computation for a general n. See also Example 10.21 p. 249-250 in Bóna's textbook.)

3. Last Section

- Email me with a few of the above exercises that you would like to present during class on Thurs Apr 23.
- Questions, comments, suggestions?