Cover Sheet (Math3250 Combinatorics)

Your Names: NAMES

Assignment: Portfolio Problems 2

In this class, you are encouraged to discuss the problems with anyone. You are also allowed to use any resources as long as you cite them.

Please cite the individuals and documents that have helped you in completing this assignment. The individuals cited should include all classmates with whom you have collaborated (even if they only act as a sounding board by listening to you during class group work). Documents cited should include Bóna's textbook and all web content you have consulted.

Individuals:

Documents:

MATH3250 PORTFOLIO PROBLEMS 2

Complete at least five of the following sections (the last section counts as one).

Overleaf Setup. To upload this template file, you can first download it to your local machine and then upload it (by clicking the upload button on the top left). Alternatively, you can create a new blank file (click on the button shaped like paper on the top left), name it "main.tex", then copy and paste the content of this template file to "main.tex".

Reference Chapter 8 of Bóna's "A Walk Through Combinatorics" textbook.

Contents

1. Sec 8.1.1: OGF from recurrence relation

Consider the sequence defined recursively by $r_0 = 3$, $r_1 = 4$, and $r_n = r_{n-1} + 6r_{n-2}$, for $n \geq 2$. Find a closed form expression for the ordinary generating function $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and use this to find a closed form expression for r_n itself.

Answer. Enter your answer here. □

2. Sec 8.1.1: OGF Tower of Lucas

In the "Tower of Hanoi" puzzle, you begin with a pyramid of n disks stacked around a center pole, with the disks arranged from the largest diameter on the bottom to the smallest diameter on top. There are also two empty poles that can accept disks. The object of the puzzle is to move the entire stack of disks to one of the other poles, subject to three constraints:

- Only one disk may be moved at a time.
- Disks can be placed only on one of the other three poles.
- A larger disk cannot be placed on a smaller one.

Let $a_0 = 0$ and let a_n be the number of moves required to move the entire stack of n disks to another pole.

a. Clearly, $a_1 = 1$. To move n disks, we must first move the $n - 1$ top disks to one of the other poles, then move the bottom disk to the third pole, then move the stack of $n - 1$ disks to that pole. Use this logic to write down a recurrence relation for a_n

b. Compute the ordinary generating function (OGF) of a_n.

Answer. Enter your answer here. □

3. Sec 8.2.1: EGF Harmonic

a. Compute by hand the Taylor series for $\ln\left(\frac{1}{1-x}\right)$ centered at 0. Read your Calculus textbook or watch lecture videos explaining this specific problem.

Hints:
i. Start with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
ii. Integrate both sides. Don't forget to compute the constant "C" has to be.
iii. Note that $\ln\left(\frac{1}{1-x}\right) = \ln(1) - \ln(1-x)$
iv. Check your answer with a computing tool.

b. Let $h_1 = 0$ and

$$h_{n+1} = (n+1)\ h_n + n! \ \text{ for all } n \geq 0.$$

Let $H(x) = \sum_{n=1}^{\infty} h_n \frac{x^n}{n!}$ be the exponential generating function of h_n. Compute an explicit formula for the exponential generating function of $H(x)$.

Hints:
Follow the examples of Section 8.2.1 Recurrence Relation and *Exponential* Generating Function.
i. Multiply both sides of the recurrence relation by $x^{n+1}/(n+1)!$ and sum over all $n \geq 0$.
ii. Let $H(x)$ be the exponential generating function for h_n. Show that $H(x) = \frac{1}{1-x} \ln\left(\frac{1}{1-x}\right)$.

c. Use the exponential generating function to prove the formula

$$h_n = n! \sum_{k=1}^{n} \frac{1}{k}.$$

Hints:
i. Write the product $H(x) = \frac{1}{1-x} \cdot \ln\left(\frac{1}{1-x}\right)$ as the product of two power series.
ii. Use Lemma 8.4 (*ordinary* generating function) to write $H(x)$ as one power series.
iii. The "extra" $n!$ factor is because $H(x) = \sum_{n=1}^{\infty} h_n \frac{x^n}{n!}$.

Answer. a. The Taylor series for $\ln\left(\frac{1}{1-x}\right)$ is

$$\boxed{\sum_{n=}^{\infty} \textit{insert } x^n}$$

b. An explicit formula for the exponential generating function is

$$\boxed{\textit{insert formula in terms of } x}.$$

c. An explicit formula for a_n is $\boxed{\textit{insert formula in terms of } n}$. □

4. Sec 8.2.1: EGF Permutations

Let $d_0 = 1$. If $n \geq 1$, let d_n be the number of bijections on $[n]$ such that, for all $i \in [n]$, it sends i to another number $j \in [n]$. For example, $d_1 = 0$, $d_2 = 1$, $d_3 = 2$.
Then d_n satisfies the recurrence (you are not asked to prove this)

$$d_n = n\ d_{n-1} + (-1)^n \text{ for } n \geq 1. \tag{1}$$

You should verify that the recurrence works for $n = 1, 2, 3$.

a. Use the recurrence relations (1) to compute the *exponential* generating function $D(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}$ for d_n.

(Hint: Follow the examples in Sec 8.2.1 Recurrence Relations and Exponential Generating Functions).

b. Use generating function method and the previous part to prove that

$$d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

You should first verify that this formula works for $n = 1, 2, 3$.

Hints:

i. In part (a), you've shown that $D(x)$ is equal to the product of two functions of x. Write this product as a product of two power series.
ii. Use Lemma 8.4 (*ordinary* generating function) to write $D(x)$ as one power series.

Answer. Enter your answer here. □

5. Sec 8.2.2: EGF The product formula

Given a classroom with n students ($n \geq 0$), a teacher divides the students into three groups A, B, C so that A has an odd number of people and B has an even number of people, and the number of people in C is a multiple of three (note: groups can be empty). The teacher then asks each group to form a line.

Let f_n be the number of ways to do this. Find its exponential generating function $F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$.

Hints:

This is very similar to Example 8.8 (pg 172) in Section 8.1.2 problem. Define three sequences, the number of ways line up n people (if n is odd), the number of ways to line up n people (if n is even), and the number of ways to line up n people (if n is a multiple of three). We have to use the *exponential* (as opposed to ordinary) generating function product formula because a group is not an interval (unlike an interval of semester days).

Answer. Enter your answer here. □

6. Sec 8.2.2: EGF the product formula

Let $f(n)$ be the number of ways to do the following. There are n (distinguishable, off course) children in a classroom. You give an odd number of the children either a red candy or a turquoise candy to eat. I give an odd number of the children either a black marble, a purple marble, or a green marble. The remaining children get nothing. (No child receives more than one item.)

For instance, f(1) = 0 (since there must be at least one child that gets a candy and at least one other child that gets a marble) and f(2) = 12 (two choices for which child gets a candy, two choices for the candy color, and three choices for the marble color).

a. Find a simple formula for the exponential generating function $F(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!}$ not involving any summation symbols.
b. Find a simple formula for f(n) not involving any summation symbols.

Answer. Enter your answer here. □

7. Collaboration, etc

i. Briefly share your group's work (at least one problem) with another group. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. An email discussion is fine if you don't have time to interact virtually in real-time.
ii. Approximately how much time did you spend on this homework?
iii. Comments and questions?

Answer. Enter your answers here. □