Math3250 Combinatorics Exam 2 Practice

Assume all graphs are simple.

1. Ref HHM Sec 1.1.2 basics

(1) What is the maximum number of edges in a graph on n vertices? What is the minimum number of edges in a graph?
(2) What is the number of possible graphs with n vertices?

(3) What is the degree sequence of a graph?
(4) Can a degree sequence have all distinct entries? Explain.

(5) What is a directed graph? (Other names are digraph and oriented graph)
(6) Is it possible make all streets one-way so that you can never return to a point you have left? Explain. Give an example with a specific graph.

(7) State the "First Theorem of Graph Theory".
(8) Prove the "First Theorem of Graph Theory".
(9) Is it true that the number of living people having an odd number of friends is even? (Define friendship to be mutual, and you are not friend with yourself.) Explain.

2. Ref HHM Sec 1.1.3 Special types of graphs

a. What does it mean for a graph to be regular?
b. Prove: If the complete bipartite $K_{r,s}$ is regular, then $r = s$.
c. HHM Sec 1.1.3, p.17: Exercise 10 (isomorphic/non-isomorphic)
d. Show me two connected graphs (both with 5 vertices) which are not isomorphic.

3. HHM Sec 1.2.1 Radius and diameter exercises

Say the definition of radius & diameter of a graph. Then find the radius and diameter of the following graphs:

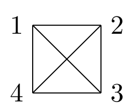
A. the graph on HHM, p.18 (Sec 1.2.1)
B. the graph on HHM, p.20 (Sec 1.2.1 Ex 1)
C. the path graph P_{2k} and P_{2k+1}
D. the cycle graph C_{2k+1} and C_{2k+1}
E. the complete bipartite graph $K_{m,n}$
F. the complete graph K_n

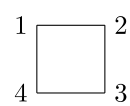
4. Sec 1.2.2 Adjacency matrix, distance matrix

Write down the definition of *adjacency matrix* and *distance matrix*. Give the adjacency matrix and the distance matrix of each of the following families of graphs:

a.) the path graph P_n, where the vertices are labeled from one end of the path to the other.
b.) the cycle graph C_{2k} and C_{2k+1}, where the vertices are labeled consecutively around the cycle, *e.g.*
c.) the complete bipartite graph $K_{m,n}$, where the vertices in the first partite set are labeled $1, 2, \ldots, m$.
d.) the complete graph K_n, any labeling.

e.) State Theorem 1.7 in HHM. You can have this ready in front of you prior to the exam.
f.) Without computing the matrix directly, find A^3 where A is the adjacency matrix of the complete graph K_4 (shown below). Use Theorem 1.7 in HHM. Recall that $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ since there is an edge between every pair of vertices.

$K_4 =$ 

$C_4 =$ 

g.) Without computing the matrix directly, find A^3 where A is the adjacency matrix of the cycle graph C_4 (shown above). Use Theorem 1.7 in HHM.
h.) If A is the adjacency matrix for a graph G, show that the (j, j) entry of A^2 is the degree of v_j.

5. Graph and generating functions

(1) How many edges does the path graph P_n have?
(2) Find a closed formula for the ordinary generating function for the number of edges of the path graph P_n, $n \geq 1$. Sanity check: Check that when you expand this function as a power series, the coefficient for x and x^2 is the number of edges of P_1, which is 0, and P_2, which is 1.

(3) How many edges does the cycle graph C_n have?
(4) Find a closed formula for the ordinary generating function for the number of edges of the cycle graph C_n, $n \geq 3$. Sanity check: Check that when you expand this function as a power series, the coefficient for x^3 and x^4 is the number of edges of C_3 and C_4.
(5) Find a closed formula for the exponential generating function for the number of edges of the cycle graph C_n, $n \geq 3$.

(6) How many edges does the complete graph K_n have?
(7) Find a closed formula for the ordinary generating function for the number of edges of the complete graph K_n, $n \geq 1$.
(8) Find a closed formula for the exponential generating function for the number of edges of the complete graph K_n, $n \geq 1$.

(9) How many edges does the complete bipartite graph $K_{n,m}$ have?
(10) Find a closed formula for the ordinary generating function for the number of edges of the complete bipartite graph $K_{n,5}$, $n \geq 1$. Check that the coefficient for x and x^2 is the number of edges of $K_{1,5}$ and $K_{2,5}$.
(11) Find a closed formula for the exponential generating function for the number of edges of the complete graph $K_{n,5}$, $n \geq 1$. Check that the coefficient for $x/1!$ and $x^2/2!$ is the number of edges of $K_{1,5}$ and $K_{2,5}$.

6. HHM Sec 1.2.3 Graph models

Talk briefly, for about one minute, about one way to use graphs to model the world (e.g., you can use HHM Sec 1.2.3)

7. From Bona: From recurrence relation to ordinary generating function (OGF)

(1) What is the *ordinary generating function* of a sequence?
(2) The frog population of a lake grows four times each year. On the first day of each year, 100 frogs are taken out of the lake. Assume that there were 50 frogs in the lake in 2020, let a_n be the number of frogs in the lake after n years. Write down a recurrence relation for a_n then find the ordinary generating function of a_n. (See solution p. 163–165)
(3) Example 8.2 (p. 167)
(4) Example 8.3 (p. 168)

8. Product Formula for Ordinary Generating Function (OGF)

See Bona, p. 169–172 Sec 8.1.2 Products of Ordinary Generating Function.

(1) Solve Example 8.6 using the Product Formula for Ordinary Generating Function.
(2) Solve Example 8.7 using the Product Formula for Ordinary Generating Function.
(3) Solve Example 8.8 using the Product Formula for Ordinary Generating Function.
(4) Give a combinatorial proof for the formula in the solution of Example 8.6. (See solution to Ch 8 Exercise 13, p. 200)
(5) Give a combinatorial proof for the formula in the solution of Example 8.7 (See solution to Ch 8 Exercise 14, p. 201)
(6) Section 8.1.2: The product formula Let f_n be the number of ways to pay n dollars using ten-dollar bills, five-dollar bills, and one-dollar bills only. Find the ordinary generating function $F(x) = \sum_{n=0}^{\infty} f_n\, x^n$. (Solution: See Ch 8 Exercise 8, pg 198–199).

9. Catalan numbers

Describe to me two Catalan objects (give me the objects for $n = 2$ where there are two, and $n = 3$ where there are five). Describe a bijection between your two objects. (See Reading Homework 9)

10. Sec 8.2.1 Exponential Gen. Functions (EGF)

- What is the exponential generating function of a sequence? Why is it called *exponential*?
- Example 8.17 (See lecture notes for Sec 8.2.1 video or watch the video), Example 8.19 (pg 181)

11. Bona Sec 8.2.2 Products of Exponential Gen. Functions

(1) Example 8.22 (See lecture notes for Sec 8.2.2 video or watch the video)
(2) A football coach has n players to work with at today's practice. First the coach splits the players into two *non-empty* groups, and then the coach puts the members of each group in a line. In how many different ways can all this happen?
Step-by-step outline for solving the problem:
    - Let $A(x)$ be the *exponential* generating function (EGF) enumerating the number of ways for (nonzero) players to line up. Explain why $A(x) = \frac{x}{1-x}$.
    - Let c_n be the number of ways for the coach to first splits the players into two non-empty groups then put the players of each group in a line. Let $C(x) := \sum_{n=0}^{n} c_n \frac{x^n}{n!}$. Use the Product formula for exponential generating functions to show that $C(x) = \frac{x^2}{(1-x)^2}$.
    - Start with the geometric series then differentiate to get the power series for $1/(1-x)^2$.
    - Multiple your answer with x^2 to show that $C(x) = \frac{x^2}{(1-x)^2} = \sum_{n=2}^{\infty}(n-1)x^n$.
    - Conclude that $c_n = n!(n-1)$ for $n \geq 2$.

12.1. **Sec 8.1.1: OGF from recurrence relation.** Consider the sequence defined recursively by $r_0 = 3$, $r_1 = 4$, and $r_n = r_{n-1} + 6r_{n-2}$, for $n \geq 2$. Find a closed form expression for the ordinary generating function $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and use this to find a closed form expression for r_n itself.

12.2. **Sec 8.1.1: OGF Tower of Lucas.** In the "Tower of Hanoi" puzzle, you begin with a pyramid of n disks stacked around a center pole, with the disks arranged from the largest diameter on the bottom to the smallest diameter on top. There are also two empty poles that can accept disks. The object of the puzzle is to move the entire stack of disks to one of the other poles, subject to three constraints:

- Only one disk may be moved at a time.
- Disks can be placed only on one of the other three poles.
- A larger disk cannot be placed on a smaller one.

Let $a_0 = 0$ and let a_n be the number of moves required to move the entire stack of n disks to another pole.

a. Clearly, $a_1 = 1$. To move n disks, we must first move the $n-1$ top disks to one of the other poles, then move the bottom disk to the third pole, then move the stack of $n-1$ disks to that pole. Use this logic to write down a recurrence relation for a_n

b. Compute the ordinary generating function (OGF) of a_n.

12.3. **Sec 8.2.1: EGF Harmonic.**

a. Compute by hand the Taylor series for $\ln\left(\frac{1}{1-x}\right)$ centered at 0.

b. Let $h_1 = 0$ and

$$h_{n+1} = (n+1)\ h_n + n! \ \text{ for all } n \geq 0.$$

Let $H(x) = \sum_{n=1}^{\infty} h_n \frac{x^n}{n!}$ be the exponential generating function of h_n. Compute an explicit formula for the exponential generating function of $H(x)$.

c. Use the exponential generating function to prove the formula

$$h_n = n! \sum_{k=1}^{n} \frac{1}{k}.$$

Then verify that this formula satisfies the recurrence relation for $n = 1, 2, 3$.

12.4. **Sec 8.2.1: EGF Permutations.** Let $d_0 = 1$. If $n \geq 1$, let d_n be the number of bijections on $[n]$ such that, for all $i \in [n]$, it sends i to another number $j \in [n]$. For example, $d_1 = 0$, $d_2 = 1$, $d_3 = 2$.

Then d_n satisfies the recurrence (you are not asked to prove this)

$$d_n = n\ d_{n-1} + (-1)^n \text{ for } n \geq 1. \tag{1}$$

You should verify that the recurrence works for $n = 1, 2, 3$.

a. Use the recurrence relations (1) to compute the *exponential* generating function $D(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}$ for d_n.

b. Use generating function method and the previous part to prove that

$$d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

Then verify that this formula works for $n = 1, 2, 3$.

12.5. **Sec 8.2.2: EGF The product formula.** Given a classroom with n students ($n \geq 0$), a teacher divides the students into three groups A, B, C so that A has an odd number of people and B has an even number of people, and the number of people in C is a multiple of three (note: groups can be empty). The teacher then asks each group to form a line.

Let f_n be the number of ways to do this. Find its exponential generating function $F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$.

12.6. **Sec 8.2.2: EGF the product formula.** Let $f(n)$ be the number of ways to do the following. There are n (distinguishable) children in a classroom. You give an odd number of the children either a red candy or a turquoise candy to eat. I give an odd number of the children either a black marble, a purple marble, or a green marble. The remaining children get nothing. (No child receives more than one item.)

For instance, f(1) = 0 (since there must be at least one child that gets a candy and at least one other child that gets a marble) and f(2) = 12 (two choices for which child gets a candy, two choices for the candy color, and three choices for the marble color).

a. Find a simple formula for the exponential generating function $F(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!}$ not involving any summation symbols.

b. Find a simple formula for f(n) not involving any summation symbols.