Sec 16.3 Part B

Thm 16.33 (Weisner's thm)

Let L be a lattice with minimum elt $\hat{0}$ and maximum elt $\hat{1}$.

Then, for $a \in L - \{\hat{1}\}$ ("any elt that's not the maximum),

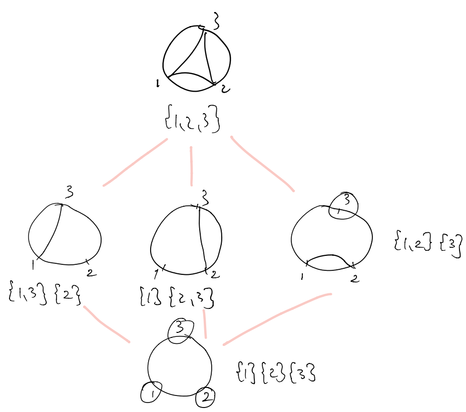
$$\mu([\hat{0},\hat{1}]) = -\sum_{\substack{x : x \wedge a = \hat{0} \\ x \neq 0}} \mu([x,\hat{1}])$$

I.e. $0 = \sum_{x : x \wedge a = \hat{0}} \mu([x,\hat{1}])$.

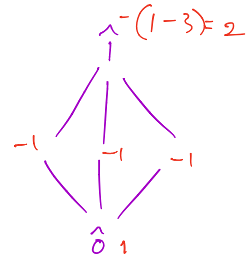
Rem Instead of computing $\mu([\hat{0},x])$ for all $x \leq \hat{1}$, we only need to compute $\mu([\hat{0},x])$ whose meet w/ a is $\hat{0}$.

"If we choose a to be a large elt, the # of these elts will be small"

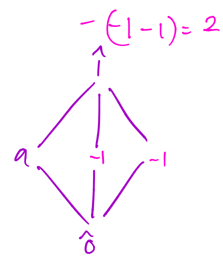
E.g. the Kreweras poset.



Before



New way



Proof ~~(if~~ time)

Recall the (lattice) poset Π_n of (set) partitions of $[n]$.

(Example 16.34)

$$\hat{1} = \{1,2,\dots,n\}$$

$$\{1\}\{2,3,\dots,n\} \qquad \{2\}\{1,3,4,\dots,n\} \qquad \cdots \qquad \{n\}\{1,2,\dots,n-1\}$$

Thm 16.33 works best if we choose $a \in P$ which is covered by $\hat{1}$.

Choose $a := \{n\}\{1,2,\dots,n-1\}$.

Then $x \wedge a = \hat{0}$ iff n and $x = \{k,n\}\{1\}\{2\}\dots\widehat{\{k\}}\dots\{n-1\}$ for some $k \in [n-1]$.

meaning $\{k\}$ is not included

Note: There are $n-1$ possibilities

If x is such an elt, then $[x,\hat{1}]$ $\widetilde{\text{isomorphic to}}$ Π_{n-1} (set) partitions of $[n-1]$

To see this, think of $[x,\hat{1}]$ as the set of partitions of $\{\{k,n\}, 1, 2, \dots, \hat{k}, \dots, n-1\}$ which is an $n-1$-elt set.

By Thm 16.33,

$$\mu_{\Pi_n}([\hat{0},\hat{1}]) = -\sum_{\substack{x:\, x\wedge a = \hat{0} \\ x \neq \hat{0}}} \mu([x,\hat{1}]) = -(n-1)\, \mu_{\Pi_{n-1}}([\hat{0},\hat{1}])$$

$\therefore$ By induction, we have

$$\boxed{\mu_{\Pi_n}([\hat{0},\hat{1}]) = (-1)^{n-1}\,(n-1)!}$$

because $\mu_{\Pi_2}([\hat{0},\hat{1}]) = -1$

Note the #s are 2, 5, 14.

Fun fact: # NC_n is the n-th Catalan #.

<u>Lemma</u> The one-block elt $\{1,2,3,\dots,n\}$ is the maximum elt of NC_n.

<u>Lemma</u> NC_n is a meet-semi lattice,

with $\underset{\text{meet}}{\alpha \wedge \beta}$ being the set partition such that

the elts i and j are in the same block in $\alpha \wedge \beta$ iff

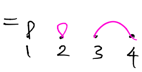
i and j are in the same block in both α and β,

i.e., i and j are connected in $\alpha \wedge \beta$ iff

i and j are connected in both α and β.

<u>Ex</u>

$\alpha =$ 
$\beta =$ 
$c =$ 

$\underset{\text{meet}}{\alpha \wedge \beta} =$  and $\underset{\text{meet}}{\alpha \wedge c} =$ 

<u>Proof</u> Let $\alpha, \beta \in NC_n$.

Let c be the partition of $[n]$ s.t

i and j are in the same block in c iff i and j are connected (in the same block) in both α and β.

$\boxed{\text{HW}}$ Prove that c is non-crossing.