Sec 16.2 Part B   Started here Week 14 Friday

Assume P is a locally finite poset (every interval is finite, e.g. $\mathbb{N}$, the set of all integer partitions)

Recall (Def 16.11) the zeta function $\zeta \in \underbrace{I(P)}_{\text{incidence algebra, the set of all functions } f:\, \underbrace{Int(P)}_{\text{nonempty intervals}} \to \mathbb{R}}$

is defined by $\zeta([x,y]) = 1$ for all nonempty intervals $[x,y]$

Recall $\underbrace{\overset{\text{delta}}{\delta}}_{\text{the "identity matrix"}}$ is the function where $\delta([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

Def 16.14

The inverse of the zeta function of P is called the Möbius function of P. Denote this μ or μ_P.   Note: $\mu \underset{\text{zeta}}{\zeta} = \underset{\text{delta}}{\delta} = \zeta\mu$.

Thm 16.15

① $\mu([x,x]) = 1$ and ② $\mu([x,y]) = -\sum\limits_{x \le z < y} \mu([x,z])$ if $x < y$.

$$\left[\text{i.e. } \quad 0 = \sum_{z \in [x,y]} \mu([x,z]) \quad \text{for all } x < y. \right]$$

Pf ① Let $x \in P$.

$1 = \delta([x,x])$

$= (\mu\zeta)([x,x])$ by def of μ

$= \sum\limits_{z \in [x,x]} \mu([x,z])\,\zeta([z,x])$ by def of multiplication in $I(P)$ (the incidence alg) (see Sec 16.2 Def 16.10)

$= \mu([x,x])\,\zeta([x,x])$

$= \mu([x,x]) \cdot 1$ by def of the zeta function.

② Let $x < y$.

$0 = \delta([x,y])$ by def of the "identity" δ function

$= \mu\zeta([x,y])$ by def of μ

$$= \sum_{z \in [x,y]} \mu([x,z])\,\zeta([z,y])$$ by def of multiplication in $I(P)$ (the incidence alg) (See Sec 16.2 Def 16.10)

$$= \sum_{z \in [x,y]} \mu([x,z])$$ since $\zeta([z,y]) = 1$ for all $z \leq y$.

We've shown that the sum of $\mu([x,z])$ taken over all $z \in [x,y]$ is 0. $\boxed{\text{Thm}}$

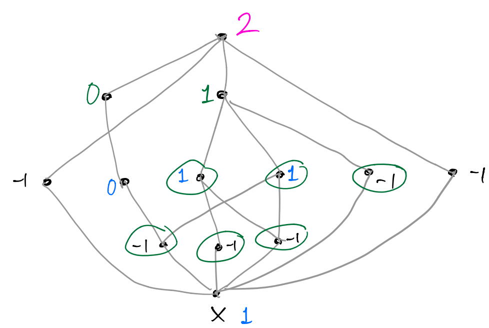
Cor If $x < y$, then

$$\mu([x,y]) = -\sum_{x < z \leq y} \mu(z,y).$$

(The same proof but use $\zeta\mu = \delta$ instead of $\mu\zeta = \delta$).

Example 16.17

Using Thm 16.15 to compute values of $\mu([x,y])$ from the bottom up.



Example 16.18

Let P be the poset of all nonnegative integers.

If $x, < y$, then

$$\mu([x,y]) = \begin{cases} -1 & \text{if } x+1 = y \\ 0 & \text{if } x+1 < y \end{cases}$$

Pf Let $x \in \mathbb{N}$

Base case: $y = x+1$ and $y = x+2$

Induction step: Suppose the statement is true all y less than $X+k$.

Prove that the statement holds for $X+k$.

HW Finish the proof.

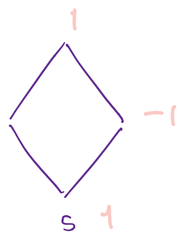
Example 16.19 $P = B_n$. If $S \subseteq T$, then

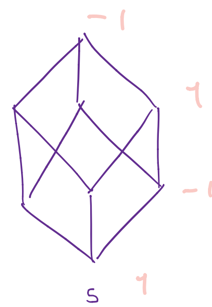
$$\mu([S,T]) = (-1)^{|T-S|}$$

(HW) Read the proof in Bona

Pf We apply strong induction on $k = |T - S|$.

compute $\mu([S,T])$





Example 16.20

$P = \mathbb{Z}_{\geq 1}$ where $x \leq y$ iff x is a factor of y.

① If $\frac{y}{x} = p_1 p_2 \cdots p_k$ is a product of distinct primes,

$\mu([x,y]) = (-1)^k$

② Otherwise (if $\frac{y}{x}$ is divisible by the square of a prime number)

$\mu([x,y]) = 0$

ended here week 14 Friday

Pf (HW) Attempt to prove ①, then read Bona's proof.

Note that the intervals $[1, \frac{y}{x}]$ and $[x, y]$ are isomorphic as posets. So it's enough to prove ② in the special case when $x = 1$.

We prove ② by strong induction on y.

First step:

$\mu([1,4]) = 0$ because $\mu([2,4]) = -\sum_{z \in (2,4] = \{4\}} \mu([z,4])$ by Thm 16.15

$= -\mu([4,4])$

$= -1$ by Thm 16.15

$\mu([1,4]) = -\sum_{z \in (1,4] = \{2,4\}} \mu([z,4])$ Thm 16.15

$$= -\mu([2,4]) - \mu([4,4])$$

$$= -(-1) - 1$$

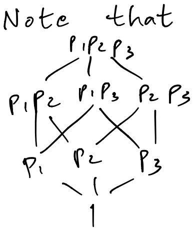
$$= 0$$

Inductive step:

Assume the statement is true for all positive integers smaller than y.

Let $p_1, p_2, \ldots, p_k$ be distinct prime divisors of y; at least one of them occurs in the prime factorization of y more than once. Call a divisor of y <u>good</u> if it is not divisible by the square of a prime. Call a divisor of y <u>bad</u> if it is divisible by the square of a prime.

Note that $\{\text{good integers } z\} = [1, p_1 p_2 \cdots p_k]$



So
$$\left.\begin{aligned} \sum_{z \text{ good}} \mu([1,z]) &= \sum_{z \in [1, p_1 \cdots p_k]} \mu([1, p_1 p_2 \cdots p_k]) \\ &= 0 \quad \text{by Thm 16.5} \end{aligned}\right\} (*)$$

$$\sum_{\substack{z \text{ bad} \\ z \neq y}} \mu([1,z]) \overset{(**)}{=} 0$$ because $\mu([1,z])$ for all bad integers by the induction hypothesis

Then
$$\mu(y) = -\sum_{1 \le z < y} \mu([1,z]) \quad \text{by Thm 16.15}$$

$$= -\sum_{z \text{ good}} \mu([1,z]) - \sum_{z \text{ bad}} \mu([1,z])$$

$$= -0 - 0 \quad \text{by } (*) \text{ and } (**)$$

$$= 0, \text{ as needed} \quad \square$$

ended here Week 15 Monday

Start here Week 15 Fri (last day)

Why do we care about the Möbius function?

Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of real #s.

Define $b_n := \sum_{i=0}^{n} a_i$ for $n \geq 0$.

Note: Given the b_i, we can compute the a_i by $a_n = b_n - b_{n-1}$.

Thm 16.21 Möbius Inversion Formula

Let $f : P \to \mathbb{R}$ be a function.

Let $g : P \to \mathbb{R}$ be defined by

$$g(y) = \sum_{x \leq y} f(x).$$

Then $f(y) = \sum_{x \leq y} g(x)\, \mu([x,y])$

ended here

skip proof

Proof

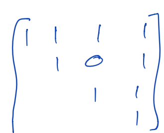
Let $x_1, x_2, \ldots, x_n$ be a linear extension of P.

E.g.

Let $\bar{f} = [f(x_1), f(x_2), \ldots, f(x_n)]$

$\bar{g} = [g(x_1), g(x_2), \ldots, g(x_n)]$

Let Z be the zeta matrix of P



and let M be the Möbius matrix of P (the inverse of Z).

Then $\bar{f} Z = [f(x_1), f(x_2), \ldots, f(x_n)]\, Z$

$= [g(x_1), g(x_2), \ldots, g(x_n)]$ because $g(y) = \sum_{x \leq y} f(x)$

$= \bar{g}$

Then $\bar{f} Z M = \bar{g} M$

so $\bar{f} = \bar{g} M$ since $ZM = I$ by def of M

Hence $f(y) = \sum_{x \leq y} g(x)\, \mu([x,y])$. $\square$

Ref: The Early (and Peculiar) History of the Möbius Fun

An application of μ

Problem: For $|x|<1$, compute

$$S(x) = \frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} - \frac{x^7}{1-x^7} + \frac{x^{10}}{1-x^{10}}$$

$$-\frac{x^{11}}{1-x^{11}} - \frac{x^{13}}{1-x^{13}} + \frac{x^{14}}{1-x^{14}} + \frac{x^{15}}{1-x^{15}} - \frac{x^{17}}{1-x^{17}} - \cdots$$

Answer

Recall $\mu([1,1]) = 1$

$$\mu([1,y]) = \begin{cases} (-1)^k & \text{is } y \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{if } y \text{ is divisible by the square of a prime.} \end{cases}$$

Write $\mu(y) := \mu([1,y])$ (for simplicity)

Note: $\mu(1) = 1$, $\mu(2) = \mu(3) = \mu(5) = \cdots = \mu(17) = (-1)^1$

$\mu(6) = \mu(10) = \mu(14) = \mu(15) = (-1)^2$

$\mu(8) = \mu(12) = \mu(16) = 0$

Let $f(x) = x + x^2 + x^3 + x^4 + \cdots$

$= \frac{x}{1-x}$ for $|x| < 1$.

We can rewrite $S(x) = \sum_{k=1}^{\infty} \mu(k) f(x^k)$