Sec 16.3 Lattices — Week 13 Wed started here —

Def Let P be a poset, & let $a, b, x, y \in P$.

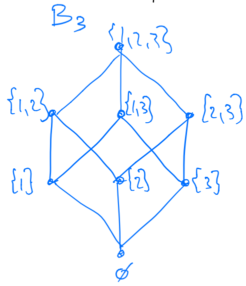
- a is an <u>upper bound for x</u> iff $x \le a$
- b is a <u>lower bound for x</u> iff $b \le x$

- a is a <u>common upper bound for x and y</u> iff
  a is an upper bound for both x and y
- b is a <u>common lower bound for x and y</u> iff
  b is a lower bound for both x and y

- a is <u>the minimum common upper bound</u> (or sup) or <u>join</u> of x & y
  iff a is smaller than or equal to every upper bound of x & y,
  written $a = x \vee y$

- b is <u>the maximum common lower bound</u> (or inf) or <u>meet</u> of x & y
  iff b is larger than or equal to every lower bound of x & y,
  written $b = x \wedge y$

B_3

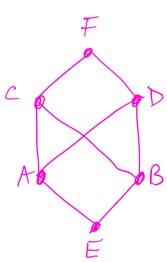


{2} has 4 upper bounds, including itself
{2} has 2 lower bounds, " "

$\{2\} \underset{\text{join}}{\vee} \{3\} = \{2,3\}$

$\{2\} \underset{\text{meet}}{\wedge} \{3\} = \emptyset$

$\{2\} \wedge \{2\} = \{2\}$



A has 4 upper bounds: A, C, D, F

B has 4 upper bounds: B, C, D, F

A and B does not have a minimum common upper bound (join),

$A \underset{\text{join}}{\vee} B$ does not exist.

$A \underset{\text{meet}}{\wedge} B = E$

$C \underset{\text{meet}}{\wedge} D$ does not exist.

Def A poset L is called a <u>lattice</u> if any two elements x and y of L have a join $x \vee y$ (sup) and a meet $x \wedge y$ (inf).

Prop The poset B_n is a lattice.

Pf For any two subsets $S \subseteq [n]$ and $T \subseteq [n]$, the minimum subset of $[n]$ containing both S and T is $S \cup T$, so $S \cup T = S \vee T$.

Similarly, $S \cap T = S \wedge T$.

Def (Exercise 12, 13)

A lattice L is <u>distributive</u> iff for all $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Prop The lattice B_n is distributive (Exercise 25)

Proof Let x, y, z be subsets of $[n]$.

Since $S \cup T = S \vee T$ and $S \cap T = S \wedge T$ $\forall$ $S, T \subseteq [n]$, we only need to show

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z).$$

But $t \in x \cup (y \cap z)$

iff $t \in x$ or ($t \in y$ and $t \in z$),

iff ($t \in x$ or $t \in y$) and ($t \in x$ or $t \in z$). (work on HW)

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Remark

The poset of all finite subsets of $\mathbb{Z}_{\geq 1}$ ordered by inclusion is a lattice where $S \vee T = S \cup T$ and $S \wedge T = S \cap T$.

This lattice has no maximal element.

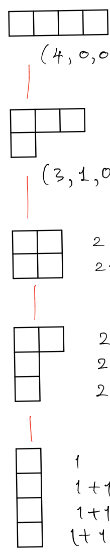
From Ch 16 Exercise (#48)

Let $n \in \mathbb{Z}_{\geq 1}$

Define a partial order (called the dominance order) on the set of all partitions of n :

We say $a = (a_1, a_2, \ldots, a_k) \leq b = (b_1, b_2, \ldots, b_t)$ iff

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + \cdots + a_k \leq \sum_{i=1}^{k} b_i = b_1 + b_2 + \cdots + b_k \text{ for all } k \geq 1.$$



$(4,0,0,0)$ 4 is bigger than every partition of 4.

$(3,1,0,0)$

$3 = 3$
$3+1 = 4$

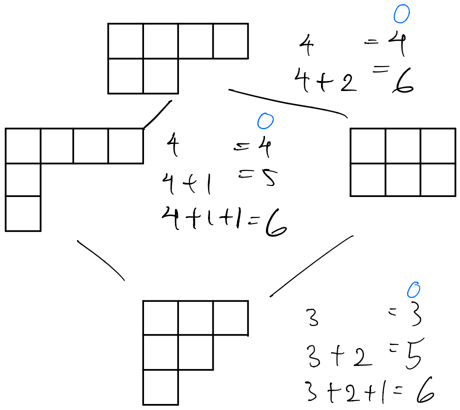
$2 = 2$
$2+2 = 4$

$2 = 2$
$2+1 = 3$
$2+1+1 = 4$

$1 = 1$
$1+1 = 2$
$1+1+1 = 3$
$1+1+1+1 = 4$

HW

Guess what the meet and join of two integer partitions should be.



$4 = 4$
$4+2 = 6$

$4 = 4$
$4+1 = 5$
$4+1+1 = 6$

$3 = 3$
$3+3 = 6$

$3 = 3$
$3+2 = 5$
$3+2+1 = 6$

$\min(a_1+a_2,\ b_1+b_2) - a_1$
$\min(a_1+a_2+a_3,\ b_1+b_2+b_3) - a_2 - a_1$

Q: Is this a lattice?

Def. A poset L is called meet-semilattice if,
for any two elts x and y of L, the meet $x \wedge y$ exists.
" — " — join-semilattice if,
— " — , the join $x \vee y$ exists.

Prop 16.29

$x, y, t \in P$ poset

① If $x \leq t$, $y \leq t$, and $\underset{\text{join}}{x \vee y}$ exists, then $x \vee y \leq t$.
② If $r \leq x$, $r \leq y$, and $\underset{\text{meet}}{x \wedge y}$ exist, then $r \leq x \wedge y$.

Proof of ② Suppose $r \leq x$ and $r \leq y$.
Then r is a common lower bound for x and y.
So r must be equal to or less than $\boxed{\text{the maximum common lower bound for } x \text{ and } y}$
which is $x \wedge y$ by def.

Lemma 16.30

Let L be a finite meet-semilattice w/ a maximal element.
Then L is a lattice.

Proof

Let $x, y \in L$. We only need to show that $\underset{\text{join}}{x \vee y}$ exists.

Let $B = \{$ all common upper bounds of x and $y \}$.

We know B is not empty because the maximum elt of L is in B.

If the minimum elt of B exists, then it is by def equal to $\underset{\text{join}}{x \vee y}$.

To show that B has a minimum elt, let $B = \{b_1, b_2, \dots, b_k\}$.

(Note: We know B is finite because L is finite.)

Then $\underset{\text{meet}}{b_1 \wedge b_2 \wedge \dots \wedge b_k}$ exists (since L is a meet-semilattice).

Denote this elt by b. (We want to show that $b \in B$.)

Since $x \leq b_1$, $x \leq b_2$, …, $x \leq b_k$,
we have $x \leq b_1 \wedge b_2 \wedge \dots \wedge b_k =: b$ by Prop 16.29②
Similarly, since $y \leq b_i$ for all $i = 1, 2 \dots, k$
we have $y \leq b_1 \wedge b_2 \wedge \dots \wedge b_k =: b$

Hence $b \in B$.
Since $b \leq b_i$ $\forall$ $i = 1, 2, \dots, k$, we conclude that b is the minimum elt of B, as needed □

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Lemma: The Dominance order D_n is a meet-semilattice.

Pf of Lemma

Suppose $a=(a_1,\dots,a_s)$ and $b=(b_1,\dots,b_t)$ are partitions of n.

Let $\hat{a}_0=0$ $\qquad\qquad$ $\hat{b}_0=0$

$\hat{a}_1=a_1$ $\qquad\qquad$ $\hat{b}_k=b_1+b_2+\dots+b_k$ for all $k=1,\dots,n$

$\hat{a}_2=a_1+a_2$ $\qquad\qquad$ note $\hat{b}_n=n$

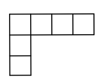
$\vdots$

$\hat{a}_n=a_1+a_2+\dots+a_n \overset{\text{note}}{=} n$

and $\hat{a}=(\hat{a}_0,\hat{a}_1,\dots,\hat{a}_n)$, $\hat{b}=(\hat{b}_0,\hat{b}_1,\dots,\hat{b}_n)$

Note: to go from $(\hat{c}_0,\hat{c}_1,\dots,\hat{c}_n)$ to $(c_1,\dots,c_n)$,
let $c_1=\hat{c}_1-\hat{c}_0$, $c_2=\hat{c}_2-\hat{c}_1$, $\dots$, $c_k=\hat{c}_k-\hat{c}_{k-1}$

Example:



4
1
1

$\hat{c}_0=0$
$\hat{c}_1=4$
$\hat{c}_2=5$
$\hat{c}_3=6$
6
6
$\hat{c}_6=6$

Let $\hat{c}:=(\hat{c}_0,\hat{c}_1,\dots,\hat{c}_n)$ where $\hat{c}_k=\min(\hat{a}_k,\hat{b}_k)$ for all $k=0,1,\dots,n$.

Claim:
The corresponding sequence $c=(c_1,\dots,c_n)$ where $c_k=\hat{c}_k-\hat{c}_{k-1}$ for all $k=1,\dots,n$
is a partition of n.

Proof of claim

- Need to show $c_1+\dots+c_n=n$:

$$\begin{aligned} c_1+c_2+\dots+c_n &= (\hat{c}_1-0)+(\hat{c}_2-\hat{c}_1)+(\hat{c}_3-\hat{c}_2)+\dots+(\hat{c}_n-\hat{c}_{n-1}) \\ &= \hat{c}_n \\ &= \min(\hat{a}_n,\hat{b}_n) \\ &= n \end{aligned}$$

because $\hat{a}_n=\hat{b}_n=n$ as we noted earlier.

• Three lemmas for showing $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_n$:

Lemma 1: $\left.\begin{array}{l} 2\hat{a}_k \geqslant \hat{a}_{k-1} + \hat{a}_{k+1} \\ 2\hat{b}_k \geqslant \hat{b}_{k-1} + \hat{b}_{k+1} \end{array}\right\}$ for all $k = 1, 2, \ldots, n-1$ (and the same statement for $\hat{b}_k$)

Pf: $2(a_1 + a_2 + \cdots + a_{k-1}) + 2a_k \geqslant 2(a_1 + a_2 + \cdots + a_{k-1}) + a_k + a_{k+1}$

since $a_k \geqslant a_{k+1}$ (because a is a partition so the sequence is non-increasing)

Lemma 2: $\hat{c}_k \leq \hat{c}_{k+1}$ for all $k = 0, 1, \ldots, n-1$

Pf: $\hat{c}_k = \min(\hat{a}_k, \hat{b}_k) \leq \min(\hat{a}_{k+1}, \hat{b}_{k+1})$ because $\hat{a}_k \leq \hat{a}_{k+1}$, $\hat{b}_k \leq \hat{b}_{k+1}$

$= \hat{c}_{k+1}$

Lemma 4: $\min(x+y, z+w) \geqslant \min(x,z) + \min(y,w)$ if $x, y, z, w \geqslant 0$

$$\min(x+y, z+w) = \frac{x+y+z+w-|x+y-z-w|}{2}$$

$$\geqslant \frac{x+y+z+w-|x-z|-|y-w|}{2}$$

$$= \min(x,z) + \min(y,w)$$

by triangle inequality $|x-z+y-w| \leq |x-z| + |y-w|$

Lemma 3: $2\hat{c}_k \geqslant \hat{c}_{k-1} + \hat{c}_{k+1}$ for all $k = 1, 2, \ldots, n-1$

Pf

$$\begin{aligned} 2\hat{c}_k &= 2\min(\hat{a}_k, \hat{b}_k) \\ &= \min(2\hat{a}_k, 2\hat{b}_k) \\ &\geqslant \min(\hat{a}_{k-1} + \hat{a}_{k+1}, \hat{b}_{k-1} + \hat{b}_{k+1}) && \text{due to Lemma 1} \\ &\geqslant \min(\hat{a}_{k-1}, \hat{b}_{k-1}) + \min(\hat{a}_{k+1}, \hat{b}_{k+1}) && \text{by Lemma 4} \\ &= \hat{c}_{k-1} + \hat{c}_{k+1} \end{aligned}$$

To show $c_1 \geqslant c_2$:

$2c_1 = 2\hat{c}_1 \underset{\text{Lemma 3}}{\geqslant} 0 + \hat{c}_2 = c_1 + c_2$

so $c_1 \geqslant c_2$

To show $c_2 \geqslant c_3$:

$2[c_1 + c_2] = 2\hat{c}_2 \geqslant \hat{c}_1 + \hat{c}_3 = c_1 + c_1 + c_2 + c_3$

so $c_2 \geqslant c_3$

To show $c_k \geqslant c_{k+1}$ for all $k = 1, \ldots, n-1$:

$\hat{c}_k + \hat{c}_{k-1} + c_k = 2\hat{c}_k \geqslant \hat{c}_{k-1} + \hat{c}_{k+1} = \hat{c}_{k-1} + \hat{c}_k + c_{k+1}$

So $c_k \geqslant c_{k+1}$

$\therefore$ we have shown that $c = (c_1, \ldots, c_n)$ is a partition of n

(Note: the same idea doesn't work for join and max)

Then c is a common lower bound of a and b

because $c_1 + c_2 + \cdots + c_k \overset{\text{def}}{=} \hat{c}_k = \min(\hat{a}_k, \hat{b}_k) \le \hat{a}_k \overset{\text{def}}{=} a_1 + a_2 + \cdots + a_k$

$\le \hat{b}_k \overset{\text{def}}{=} b_1 + b_2 + \cdots + b_k$ } for all k

It is also the greatest common lower bound of a and b.
To see this, let $d = (d_1, d_2, \ldots, d_n)$ be a common lower bound for a and b. Let $\hat{d} = (\hat{d}_0, \hat{d}_1, \ldots, \hat{d}_n)$ where

$$\hat{d}_0 = 0$$
$$\hat{d}_k = d_1 + \cdots + d_k \quad \text{for all } k = 1, \ldots, n$$
$$\hat{d}_n = n.$$

Then $\hat{d}_k \le \hat{a}_k$, $\hat{d}_k \le \hat{b}_k$ for all $k = 1, \ldots, n$

So $\hat{d}_k \le \min(\hat{a}_k, \hat{b}_k) = \hat{c}_k$ for all $k = 1, \ldots, n$

By def (of dominance order), $d \le c$. $\square$

Thm (Exercise #48)
The Dominance order D_n is a lattice.

Proof
The set D_n is finite because there are finitely many partitions of n.

Claim The partition (n) is the maximum element of D_n.

Pf of claim: Suppose $(a_1, a_2, \ldots, a_k)$ is a partition of n.

Then
$$a_1 \le n$$
$$a_1 + a_2 \le n$$
$$\vdots$$
$$a_1 + a_2 + \cdots + a_k \le n$$

The previous Lemma says that D_n is a meet-semi lattice.

Lemma 16.30 that any finite meet-semi lattice with a maximum elt is a lattice

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$\boxed{\text{Thm}}$

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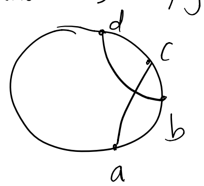
Def (Ch 14 Exercise #15 pg 365)

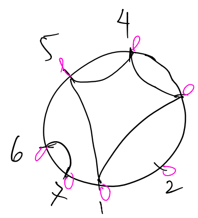
A (set) partition π of $[n]$ having blocks $\beta_1, \beta_2, \ldots, \beta_k$ is called <u>non-crossing</u> iff:

there are no four elts $1 \le a < b < c < d \le n$ so that $a, c \in \beta_i$ and $b, d \in \beta_j$ for some distinct blocks β_i and β_j.

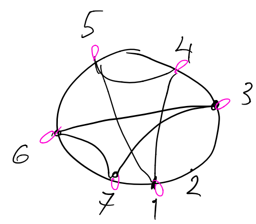
I.e. no

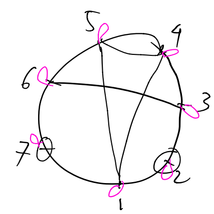


E.g.



$\{1,3,4,5\}\{2\}\{6,7\}$ is a non-crossing partition of $[6]$





$\{1,4,5\}\{2\}\{3,6,7\}$ and $\{1,4,5\}\{2\}\{3,6\}\{7\}$ are <u>not</u> non-crossing partitions of $[6]$.

Let NC_n denote the set of all non-crossing partitions of $[n]$.

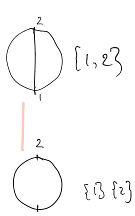
Put the refinement order (Example 16.6) on NC_n:

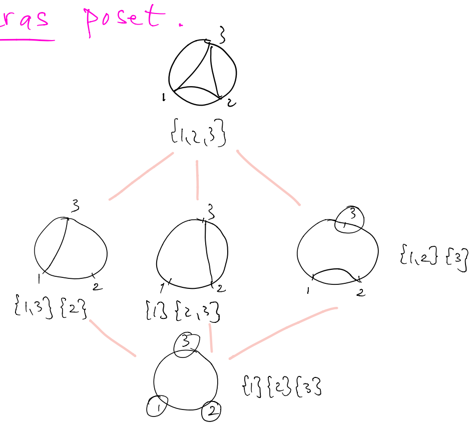
Say $\alpha \le \beta$ iff each block of β is a union of some blocks of α.

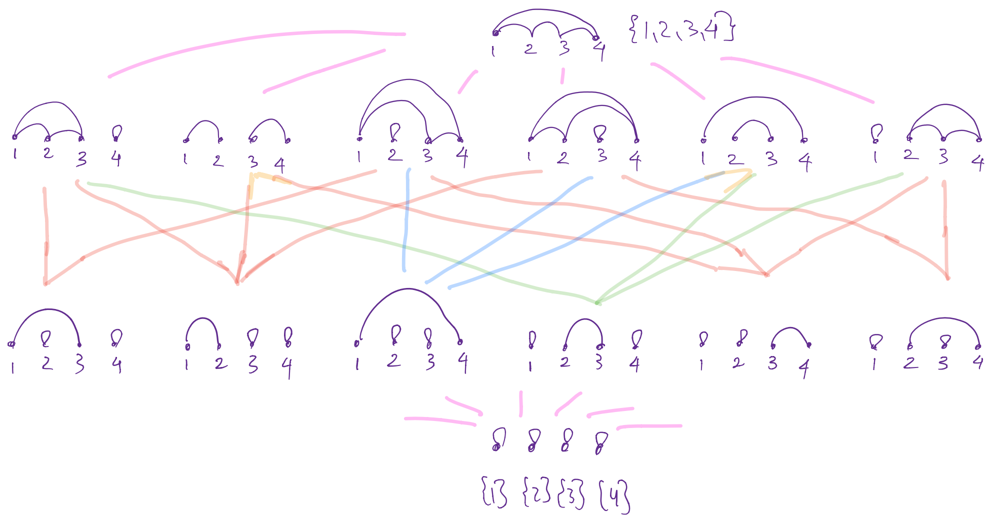
I.e. $\alpha \le \beta$ iff [if i and j are connected in α, then i and j are connected in β]

This is called the <u>Kreweras poset</u>.

<u>Ex</u> $n = 2, 3, 4$







Note the #s are 2, 5, 14.

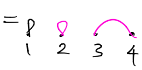
Fun fact: $\# NC_n$ is the n-th Catalan #.

Lemma The one-block elt $\{1,2,3,\dots,n\}$ is the maximum elt of NC_n.

Lemma NC_n is a meet-semi lattice,
with $\underset{\text{meet}}{\alpha \wedge \beta}$ being the set partition such that
the elts i and j are in the same block in $\alpha \wedge \beta$ iff
i and j are in the same block in both α and β,
i.e., i and j are connected in $\alpha \wedge \beta$ iff
i and j are connected in both α and β.

Ex

$\alpha =$  $\beta =$  $c =$ 

$\underset{\text{meet}}{\alpha \wedge \beta} =$  and $\underset{\text{meet}}{\alpha \wedge c} =$ 

Proof Let $\alpha, \beta \in NC_n$.
Let c be the partition of $[n]$ s.t
i and j are in the same block in c iff i and j are connected (in the same block) in both α and β.

$\boxed{\text{HW}}$ Prove that c is non-crossing.

Then $c \leq \alpha$ (and $c \leq \beta$) because
if i and j are connected in c then i and j are connected in α (resp β),
so c is a common lower bound of α and β.

To show that c is the maximum common lower bound,
suppose $d \in NC_n$ with $d \leq \alpha$, $d \leq \beta$.

To show that $d \leq c$, we need to show:
[If i and j are connected in d,
then i and j are connected in c].

Suppose i and j are connected in d.

Then i and j are connected in α and β (since $d \leq \alpha, d \leq \beta$).

Then, by def of c, we must have i and j be connected in c.

("We've shown that c is the meet of x and y") $\boxed{\text{end of Lemma}}$

<u>Corollary</u> NC_n is a lattice

<u>Pf</u> By the previous lemmas,
NC_n is a meet-semi lattice with a maximum element.
Since NC_n is finite, it is a lattice by Lemma 16.30.