The famous Catalan #s (Ref: 8.1.2.1 or google "Catalan #s")

Object 1: Grouping with n parentheses

$n=1$ ()

$n=2$ (()), ()()

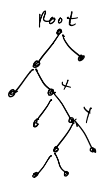
$n=3$ ((())), ()(()), (())(), (()()), ()()()

Object 2: Full binary tree with n parent vertices

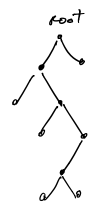
<u>Def</u> A <u>full binary tree</u> is a tree with a distinguished vertex called <u>the root</u> s.t every parent vertex has exactly two children.

E.g.

Notation: x is the parent of y, and y is a child of x

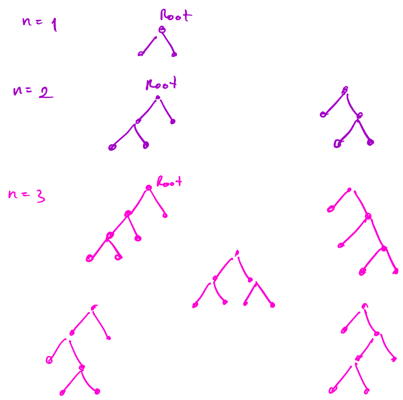


yes



NO

No because is a parent vertex but it has only one child.

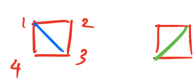


See also: "binary trees with n vertices" in Problem Set (Week 10)

Object 3: Triangulation of an (n+2)-gon

$n=1$

$n=2$



$n=3$

$n=4$

...

On board in groups: Pick your favorite objects & draw the objects for $n=4$

ended here Week 7 Friday

Problem Let $C_0 = 1$ and (Con't after presentations)

let C_n denote the # of triangulations of an $(n+2)$-gon for $n \geq 1$.

① Prove that "the rec. rel $C_n = C_0C_{n-1} + C_1C_{n-2} + \dots + C_{n-1}C_0$ for $n \geq 1$ holds"

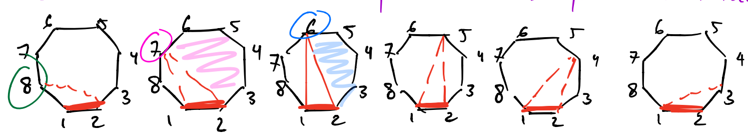
Answer

Compute $C_1 = 1$, $\triangle$ and $C_1 = C_0C_0$ is satisfied.

Let $n \geq 2$. Distinguish one side of the $(n+2)$-gon P and call it the base.

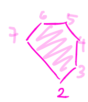
In any triangulation of P, the base forms one side of a triangle.

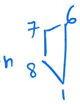
E.g. if $n=6$ there are 6 possibilities for the triangle which the base is a side of.

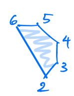


If the third corner is labeled 8, then triangulate the remaining 7-gon in C_5 ways.
$5+2$

If the third corner is labeled 7, then triangulate  in C_1 way

and triangulate the 6-gon  in C_4 ways, for a total of C_1C_4 triangulations.

If the third corner is labeled 6, then triangulate the 4-gon  in C_2 ways and

triangulate the 5-gon  in C_3 ways, for a total of C_2C_3 triangulations.

Continuing this produces $C_6 = C_5 + C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 + C_5$.

In general, the same idea gives

$$C_n = C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \dots + C_{n-2}C_1 + C_{n-1} \quad \text{for } n \geq 1.$$

Since $C_0 = 1$, we can write $C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$ □

ended here wk 8 Mon

② Use this recurrence relation to compute the generating function $F(x) = \sum_{n=0}^{\infty} C_n x^n$

Ans Multiply the recurrence relation by x^n & sum over all $n \geq 1$

$$\sum_{n=1}^{\infty} C_n x^n = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} C_k C_{n-1-k} \right) x^n \qquad \text{LHS is } F(x) - C_0 = F(x) - 1.$$

RHS is $\underbrace{C_0C_0x^1}_{\text{for } n=1} + \underbrace{(C_0C_1 + C_1C_0)x^2}_{\text{for } n=2} + \underbrace{(C_0C_2 + C_1C_1 + C_2C_0)x^3}_{\text{for } n=3} + \dots$

$= x\left[C_0C_0 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1C_1 + C_2C_0)x^2 + \dots \right]$

$= x\left[F(x)\right]^2$ since $\boxed{\left(\sum_{n=0}^{\infty} C_n x^n\right)\left(\sum_{n=0}^{\infty} C_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} C_i C_{n-i}\right)x^n \text{ by Sec 8.1.2 Lemma} \quad = C_0C_0 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1C_1 + C_2C_0)x^2 + \dots}$

$\therefore \quad F(x) - 1 = x\left[F(x)\right]^2$

$0 = x\left[F(x)\right]^2 - F(x) + 1$

+ or − ?

So $F(x) = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{1 \pm \sqrt{1-4x}}{2x}$

By def, $\sum_{n=0}^{\infty} c_n x^n \Big|_{x=0} = C_0 = 1$ "so we need F(x) to have constant term 1".

$$\lim_{x \to 0^+} \frac{1+\sqrt{1-4x}}{2x} = +\infty$$

$1+\sqrt{1-4x} \to 2$ as $x \to 0$

$2x \to 0$ as $x \to 0^+$

$$\lim_{x\to 0} \frac{1-\sqrt{1-4x}}{2x} = \lim_{x \to 0} \frac{-\frac{1}{2}(1-4x)^{-\frac{1}{2}}(-4)}{2}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{1-4x}} = 1$$

$\therefore \boxed{F(x) = \frac{1-\sqrt{1-4x}}{2x}}$ $\square$

③ Find an explicit formula for C_n.

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4)^n x^n$$ by Binomial Thm

Compute $\binom{\frac{1}{2}}{n}$:

$\binom{\frac{1}{2}}{0} = 1$, $\binom{\frac{1}{2}}{1} = \frac{1}{2}$

$\frac{1}{2} - \frac{2n}{2} + \frac{2}{2}$

$= \frac{-2n+3}{2}$

· If $n \geq 2$, then
$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}$$
$$= \frac{(-1)\cdot(-3)\cdot(-5)\cdots(-2n+3)}{2^n \; n!}$$
$$= (-1)^{n-1}\frac{(1)(3)(5)\cdots(2n-3)}{2^n \; n!}$$
$$= (-1)^{n-1}\frac{(2n-3)!!}{2^n \; n!}$$

Note: Semifactorial, $k!!$ is the product of all odd integers from 1 to $2n-3$.

$$\therefore \sqrt{1-4x} = 1 - 2x + \sum_{n\geq 2} \frac{(-1)^{n-1}(2n-3)!!}{2^n \; n!}(-4x)^n$$

$$= 1 - 2x - \sum_{n \geq 2} \frac{(2n-3)!! \; 2^n}{n!} x^n$$ because $\boxed{(-1)^{n-1}(-1)^n = (-1)^{2n-1} = -1}$ and $\frac{4^n}{2^n} = 2^n$

Rem
$$\frac{2^n(2n-3)!!}{n!}\cdot\frac{(n-1)!}{(n-1)!} = \frac{2}{n}\,\frac{(2n-3)!!\;2^{n-1}(n-1)!}{(n-1)!\,(n-1)!}$$
$$= \frac{2}{n}\cdot\frac{(2n-2)!}{(n-1)!\,(n-1)!}$$
$$= \frac{2}{n}\binom{2n-2}{n-1}$$

because $\underbrace{2\cdot 2\cdot 2 \cdots 2}_{n-1}\cdot (n-1)!$

$= 2(n-1)\cdot 2(n-2)\cdots(2)$

$$\therefore \sqrt{1-4x} = 1 - 2x - 2\sum_{n\geq 2}\frac{1}{n}\binom{2n-2}{n-1}x^n.$$

$$F(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1}{2x} - \frac{1}{2x}\left(\cancel{1} - 2x - 2\sum_{n=2}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n\right)$$
$$= 1 + \frac{1}{x}\sum_{n=2}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n$$
$$= 1 + \sum_{n=2}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^{n-1}$$
$$= 1 + \sum_{n=1}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$
$$= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$

$$\therefore \boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

n = 0, 1, 2, 3, 4, 5, 6, 7, 8, ...

1, 1, 2, 5, 14, 42, 132, 429, 1430, ...