Math3250 Combinatorics Week8 Problem Set Key to Problem 4

your preferred first and last name

1 Section 8.1.1: Recurrence relations and generating functions

2 Section 8.1.2: The product formula

3 No part divisible by three

Show that the number of partitions of n for which no part appears more than twice is equal to the number of partitions of n for which no part is divisible by 3. For instance, when $n = 5$ there are five partitions of the first type

- (5),
- $(4, 1)$,
- $(3, 2)$,
- $(3, 1, 1)$,
- $(2, 2, 1)$

and five of the second type

- (5),
- $(4, 1)$,
- $(2, 2, 1)$,
- $(2, 1, 1, 1)$,
- $(1, 1, 1, 1, 1)$.

Solution:

Proof. Let $a_0 = 1.$ and $b_0 := 1$. For $n \geq 1$, let a_n be the number of partitions of n for which no part appears more than twice and let b_n be the number of partitions of n for which no part is divisible by 3. In effort to show that $\{a_n\}$ and $\{b_n\}$ are the same sequence, we define $A(x) := \sum_{n=0}^{\infty} a_n x^n$ and $B(x) := \sum_{n=0}^{\infty} a_n x^n$.

Recall that an integer partition of n is $(p_1, p_2, p_3, \ldots, p_k)$ such that $n = p_1 + p_2 + p_3 + \cdots + p_k$ and $p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_k$. Instead of writing $n = p_1 + p_2 + p_3 + \cdots + p_k$, we can write

$$n = 1j_1 + 2j_2 + \cdots + rj_r \text{ for some } r \text{ with } j_r \geq 1, \text{ where } j_1, j_2, \ldots, j_r \in \{0, 1, \ldots, n\}.$$

An integer partition of $n \geq 1$ for which each part appears 0, 1, or 2 times can be written as $n = 1j_1 + 2j_2 + \cdots + rj_r$ with $j_r \geq 1$, where $j_1, j_2, \ldots, j_r \in \{0, 1, 2\}$. Then

$$\begin{aligned}
A(x) &= \prod_{i\geq 1}(1 + x^i + x^{i2}) \qquad (1)\\
&= \prod_{i\geq 1}(x^{i.0} + x^{i.1} + x^{i.2})\\
&= (x^{1.0} + x^{1.1} + x^{1.2})(x^{2.0} + x^{2.1} + x^{2.2})(x^{3.0} + x^{3.1} + x^{3.2})(x^{4.0} + x^{4.1} + x^{4.2})\ldots\\
&= 1 + (x^{1.1}) + (x^{1.2} + x^{1.0+2.1}) + (x^{1.1+2.1} + x^{1.0+2.0+3.0}) +\\
&(x^{1.2+2.1} + x^{1.1+3.1} + x^{1.0+2.2} + x^{1.0+2.0+3.0+4.1}) + \ldots.
\end{aligned}$$

To see why line (1) holds, note that the coefficient of x^n in the expansion of $\prod_{i\geq 1}(x^0 + x^i + x^{2i})$ is the number of ways to write n as the sum $1j_1 + 2j_2 + \cdots + rj_r$ for some r with $j_r \geq 1$, where $j_1, j_2, \ldots, j_r \in \{0, 1, \ldots, n\}$.

Next, we determine the generating function $B(x)$. Observe that an integer partition of $n \geq 1$ for which no part is divisible by 3 can be written as

$$n = \sum_{1\geq k\geq r,\ k \text{ not divisible by } 3} k\ j_k = 1j_1 + 2j_2 + 4j_4 + 5j_5 + 7j_7 + 8j_8 + 10j_{10} + \cdots + rj_r$$

with $j_r \geq 1$, where $j_1, j_2, \ldots, j_r \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}
B(x) &= \prod_{i\geq 1 \text{ and } i \text{ not divisible by } 3}(x^{i.0} + x^{i.1} + x^{i.2} + x^{i.3} + x^{i.4} + x^{i.5} + \ldots) \qquad (2)\\
&= (x^{1.0} + x^{1.1} + x^{1.2} + x^{1.3} + x^{1.4} + \ldots)(x^{2.0} + x^{2.1} + x^{2.2} + x^{2.3} + x^{2.4} + \ldots)\\
&(x^{4.1} + x^{4.2} + x^{4.3} + x^{4.4} + \ldots)(x^{5.0} + x^{5.1} + x^{5.2} + x^{5.3} + x^{5.4} + \ldots)\ldots\\
&= 1 + (x^{1.1}) + (x^{1.2} + x^{1.0+2.1}) + (x^{1.3} + x^{1.1+2.1}) +\\
&(x^{1.4} + x^{1.2+2.1} + x^{1.0+2.2} + x^{1.0+2.0+3.0+4.1}) + \ldots
\end{aligned}$$

To see why line (2) holds, note that the coefficient of x^n in the expansion of $\prod_{i\geq 1 \text{ and } i \text{ not divisible by } 3}(x^{i.0} + x^{i.1} + x^{i.2} + x^{i.3} + x^{i.4} + x^{i.5} + \ldots)$ is the number of ways to write n as the sum

$$\sum_{1\geq k\geq r,\ k \text{ not divisible by } 3} k\ j_k \quad \text{for some } r \text{ with } j_r \geq 1 \text{ with } j_k \geq 0.$$

We will now prove that $A(x) = B(x)$. We have

$$\begin{aligned}
A(x) &= \prod_{i\geq 1}(x^0 + x^i + x^{2i})\\
&= \prod_{i\geq 1} 1 - x^{3i} \prod_{i\geq 1} \frac{1}{1 - x^i} \quad \text{because } (1 + x^i + x^{2i})(1 - x^i) = 1 - x^{3i}\\
&= \prod_{i\geq 1} 1 - x^{3i} \prod_{i\geq 1} \frac{1}{1 - x^{3i}} \prod_{m\geq 0} \frac{1}{1 - x^{3m+1}} \frac{1}{1 - x^{3m+2}}\\
&= \prod_{i\geq 1} 1 - x^{3i} \prod_{i\geq 1} \frac{1}{1 - x^{3i}} \prod_{i\geq 1 \text{ and } i \text{ not divisible by } 3} \frac{1}{1 - x^i}\\
&= \prod_{i\geq 1 \text{ and } i \text{ not divisible by } 3} \frac{1}{1 - x^i}\\
&= B(x) \quad \text{since } \frac{1}{1 - t} = 1 + t + t^2 + t^3 + \ldots.
\end{aligned}$$

Hence the coefficients of $A(x)$ match the coefficients of $B(x)$, and we can conclude that $a_n = b_n$ for all $n \geq 1$.

□

4 Combinatorial proof

Copy and paste here any **one** of the problems (or subproblems) given above, and find a combinatorial proof for the result.[1]

5 Polygon diagonals (from week 4 problem set)

Let $n \geq 4$. Consider a convex n-gon that is drawn in such a way that no three diagonals intersect in one point. How many intersection points do the diagonals have? (For example, if you draw a pentagon, there are five diagonals and five crossings.) Prove this.
Uncomment for a hint:

6 Miscellaneous

i. Share your work (at least one problem) and thought process with at least one classmate. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. A virtual discussion via Piazza or email is fine if you don't have time to interact in person.

ii. Approximately how much time did you spend on this homework?

[1] If you don't come up with a combinatorial proof by the deadline, you can instead write a proof using any method (without generating functions), for example, induction.