MATH3250 COMBINATORICS WEEK8 PROBLEM SET

YOUR PREFERRED FIRST AND LAST NAME

Credit: Write down everyone who helped you, including classmates who contributed to your thought process. Write down Bona's textbook and other written sources you used as well.

Instruction. Please send me an invite via Overleaf. You are encouraged to work with other people, but this week your must write your own solution.

1. Section 8.1.1: Recurrence relations and generating functions

Type the final answers in LaTeXbelow. Because the equations may be tedious to type, you can submit handwritten work at the beginning of class (or type your work below, if you prefer). If you have to take partial fractions, you may use a computing tool (or not), but do the rest of your work by hand.

(1) Let $a_0 = 1$ and $a_{n+1} = 3a_n + 2^n$ if $n \geq 0$.
   - Find an explicit formula for the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$.
   - Use the previous part to compute an explicit formula for a_n for $a_n \geq 0$.

(2) Let $a_0 = 1, a_1 = 4$ and $a_{n+2} = 8a_{n+1} - 16a_n$ if $n \geq 0$.
   - Use the recurrence relation to find an explicit formula for the ordinary generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$.
   - Use the previous part to compute an explicit formula for a_n for $n \geq 2$.

   Uncomment for a hint:

(3) A mice population multiplies so that at the end of each year, its size is the double of its size a year before, plus 1000 more mice. Assuming that originally we released 50 mice, how many of them will we have at the end of the nth year? Use (ordinary) generating functions.

Answer. (1) An explicit formula for the ordinary generating function is

$$\boxed{insert\ formula\ in\ terms\ of\ x}.$$

An explicit formula for a_n is $\boxed{insert\ formula\ in\ terms\ of\ n}$.

(2) An explicit formula for the ordinary generating function is

$$\boxed{insert\ formula\ in\ terms\ of\ x}.$$

An explicit formula for a_n is $\boxed{insert\ formula\ in\ terms\ of\ n}$.

(3) We will have $\boxed{insert\ formula\ in\ terms\ of\ n}$ mice at the end of the nth year.

□

2. Section 8.1.2: The product formula

Let f_n be the number of ways to pay n dollars using ten-dollar bills, five-dollar bills, and one-dollar bills only. Find the ordinary generating function $F(x) = \sum_{n=0}^{\infty} f_n\, x^n$. Use it to find a closed form formula for f_n.
Uncomment for a hint:

3. All integer partitions

For $n \geq 0$, let $p(n)$ denote the number of partitions of the integer n. Prove that

$$\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Hint: Your proof should be almost the same as the proof of Example 8.9 (pg 173). Since this class is an introductory combinatorics class, your proof should be at least twice as detailed as Bona's.

4. No part divisible by three

Show that the number of partitions of n for which no part appears more than twice is equal to the number of partitions of n for which no part is divisible by 3. For instance, when $n = 5$ there are five partitions of the first type

- (5),
- $(4, 1)$,
- $(3, 2)$,
- $(3, 1, 1)$,
- $(2, 2, 1)$

and five of the second type

- (5),
- $(4, 1)$,
- $(2, 2, 1)$,
- $(2, 1, 1, 1)$,
- $(1, 1, 1, 1, 1)$.

Use generating functions.
Uncomment for hint 1: Uncomment for a hint 2:

5. Combinatorial proof

Copy and paste here any **one** of the problems (or subproblems) given above, and find a combinatorial proof for the result.[1]

6. Polygon diagonals (from week 4 problem set)

Let $n \geq 4$. Consider a convex n-gon that is drawn in such a way that no three diagonals intersect in one point. How many intersection points do the diagonals have? (For example, if you draw a pentagon, there are five diagonals and five crossings.) Prove this.
Uncomment for a hint:

[1] If you don't come up with a combinatorial proof by the deadline, you can instead write a proof using any method (without generating functions), for example, induction.

7. Miscellaneous

i. Share your work (at least one problem) and thought process with at least one classmate. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. A virtual discussion via Piazza or email is fine if you don't have time to interact in person.
ii. Approximately how much time did you spend on this homework?