Math3250 Combinatorics Week6 Exam Sample

1 Nine vectors

Let $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$, $\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ $\ldots$, $\begin{pmatrix} a_9 \\ b_9 \\ c_9 \end{pmatrix}$ be nine vectors in $\mathbb{Z}^3$. Prove that at least two of these nine vectors have a sum whose coordinates are all even integers.

Solution:

Proof. Given an integer x, let $x' = 0$ if x is even and 1 if x is odd. There are eight possible sequences of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$; hence, by the pigeonhole-principle, there two distinct $i, j \in [9]$ such that $\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix}$. Then the sum $\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} + \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} = \begin{pmatrix} a_i + a_j \\ b_j + b_j \\ c_i + c_j \end{pmatrix}$ of these two vectors have even coordinates. □

2 RIFFRAFF

How many different ways are there to arrange the letters in the word RIFFRAFF? How many different ways are there to arrange the letters in the word RIFFRAFF so that the two R's are not adjacent?

Solution:

Answer. The number of all arrangements of the multiset of 8 objects with 4 objects (that look like F), 2 objects (that look like R), one I, and 1 A is $\binom{8}{4,2,1,1}$. The number of arrangements where the two R's *are* adjacent is the same as the number of all arrangements of the multiset of 7 objects with 4 objects (that look like F), 1 object (that looks like RR), one I, and 1 A. Subtracting the second number from the first number we get $\boxed{\binom{8}{4,2,1,1} - \binom{7}{4,1,1,1}} = 4 \cdot 7 \cdot 6 \cdot 5 - \cdot 7 \cdot 6 \cdot 5 = 3 \cdot 7 \cdot 6 \cdot 5 = \boxed{630}$. □

3 Binary words

(i) Let $f(n)$ be the number of binary sequences $a_1, a_2, \ldots, a_n$ (note that this means that each a_i is 0 or 1). Note that $f(0) = 1$ because there is one binary sequence of length 0, empty sequence. Find a simple formula for $f(n)$.

Solution:

Answer. For each a_i, there are two options, so $\boxed{f(n) = 2^n}$. Read the first theorem of Section 3.2 in Bona and its proof. □

(ii) let $g(n)$ be the number of binary sequences $a_1, a_2, \ldots, a_n$ with no two consecutive 1's. Find a simple formula for $g(n)$. Note that $g(0) = 1$ because there is one binary sequence of length 0, empty sequence. Express your answer in terms of the Fibonacci numbers (given by $F_1 = F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$).

Solution: We have $g(0) = 1$, $g(1) = 2$, $g(2) = 3$, suggesting that perhaps $g(n) = F_{n+2}$.

Proof. To prove that $g(n) = F_{n+2}$, we just need to show that $g(n+1) = g(n) + g(n-1)$ for $n \geq 2$, since we already checked the initial conditions. If a sequence $a_1, \dots, a_{n+1}$ with no two consecutive 1's ends with $a_{n+1} = 0$, then $a_1, \dots, a_n$ can be any sequence of length n with no two consecutive 1's, of which there are $g(n)$. On the other hand, if $a_{n+1} = 1$, then we must have $a_n = 0$, and there are $g(n-1)$ choices for $a_1, \dots, a_{n-1}$. Thus $g(n+1) = g(n) + g(n-1)$. □

4 Compositions where each part is divisible by three

Let C be the set of compositions of 24 (into any number of parts) such that each part is divisible by 3. How many elements does C have?

Solution:

Answer. The map C to the set of all compositions of 8 defined by dividing each part of a composition in C by three is a bijection. Since the number of compositions of n is the same as the number of all subsets of $[n-1]$ (which is 2^{n-1}), the number of all compositions of 8 is 2^{8-1}. So the number of elements in C is 2^{8-1}. □

5 Bijections

Let $n \geq 4$. How many bijections $\pi : [n] \to [n]$ satisfy $\pi(1) = 2$, $\pi(2) \neq 3$, $\pi(2) \neq 4$, and $\pi(3) \neq 4$? Give a simple formula not involving summation symbols.

(Afterwards, you should check that your formula works for $n = 4$).

Solution:

Answer. There is only one choice for $\pi(1)$. There are then $n-3$ choices for $\pi(2)$ (anything other than 2, 3, and 4). There are then $n-3$ choices for $\pi(3)$ (anything other than 2, $\pi(2)$, and 4, which are all different). There are then $n-3$ choices for $\pi(4)$, $n-4$ choices for $\pi(5)$, etc. This gives $\boxed{(n-3)^3(n-4)! = (n-2)^2(n-3)!}$ choices in all. (For $n = 4$, the formula gives 1, and the only such bijection is the map sending 1 to 2, 2 to 1, and fixing both 3 and 4 pointwise. □

6 Finding an identity

Find a simple formula (no summation symbols) for

$$f(n) = \sum_{k=0}^{n} \binom{k}{2}\binom{n}{k}.$$

Solution:

Answer 1. The right-hand side counts the number of ways to choose a subset S of any size from $[n]$, then choose a 2-element subset T from S. But we could get the same result by choosing T first in $\binom{n}{2}$ ways, then choose an arbitrary subset of the remaining $n-2$ elements in 2^{n-2} ways, which gives $f(n) = \binom{n}{2}2^{n-2}$. □

Answer 2. Take the binomial expansion $(1+x)^n = \sum_{k=0}^{n} x^k$, differentiate twice and then divide both sides by 2. Then set $x = 1$ to get

$$\binom{n}{2}2^{n-2} = \sum_{k=0}^{n} \binom{k}{2}\binom{n}{k}$$

□

7 Integer partitions with no parts equal to 1

For $n \geq 2$, let $f(n)$ be the number of (integer) partitions of n with no parts equal to 1. For example, $f(1) = 0$, $f(2) = 1$ because the only such partition is (2), $f(3) = 1$ counts the partition (3), $f(4) = 2$ counts the partitions (4) and $(2, 2)$ $f(5) = 2$ counts the partitions (5) and $(3, 2)$.

Express $f(n)$ in terms of the partition function, i.e. in terms of the numbers $p(1), p(2), p(3), \ldots, p(k)$ where $p(k)$ is the number of partitions of k. Your formula should be simple, containing no summation symbols.

Solution:

Answer. We claim that $f(n) = p(n) - p(n-1)$. To obtain a partition of the integer n with at least one part equal to 1, take a partition λ of $n - 1$ and adjoin a new part equal to 1. Thus there are $p(n-1)$ with (at least) one part equal to 1, so $f(n) = p(n) = p(n-1)$. □

8 Enumerating all subsets

Given a positive integer n, what is the the number of all subsets of $[n]$?

a. Prove by induction on n.

Solution: The answer is 2^n.

Proof (by induction) from Theorem 2.4 page 27 of Bona. For $n = 1$, the statement is true as $[1] = \{1\}$ has two subsets, the empty set, and $\{1\}$.

Now let k be a positive integer, and assume that the statement is true for $n = k$. We divide the subset of $[k+1]$ into two classes: there will be those subsets that do not contain the element $k+1$, and there will be those that do. Those that do not contain $k+1$ are also subsets of $[k]$, so by the induction hypothesis their number is 2^k. Those that contain $k+1$ consist of $k+1$ and a subset of $[k]$. However, that subset of $[k]$ can be any of the 2^k subsets of $[k]$, so the number of these subsets of $[k+1]$ is once more 2^k. So altogether, $[k+1]$ has $2^k + 2^k = 2^{k+1}$ subsets, and the statement is proven. □

b. Prove by another method.

Solution: There is a proof using a bijection in Section 3.2 and there is a proof in Section 4.1 using the binomial theorem.

9 A sequence

Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and let

$$a_{n+1} = 2(a_0 + a_1 + \cdots + a_n)$$

for $n \geq 0$. Prove that $a_n = 2 \cdot 3^{n-1}$ for $n \geq 1$.

Solution:

Proof. We prove this by strong induction on n. Since $2(a_0) = 2(1) = 2 \cdot 3^{1-1}$, the initial case (for $n = 1$) is verified. Now let us assume that the statement is true for all positive integers that are less than or equal to n. Then, we have

$$\begin{aligned}
a_{n+1} &= 2(a_0 + a_1 + a_2 + \cdots + a_n) \quad \text{by the recurrence relation} \\
&= 2a_0 + 2(a_1 + a_2 + \cdots + a_n) \\
&= 2 + 2(2 \cdot 1 + 2 \cdot 3 + \cdots + 2 \cdot 3^{n-1}) \quad \text{by the induction hypothesis} \\
&= 2 + 4(1 + 3 + \cdots + 3^{n-1}) \\
&= 2 + 4\left(\frac{3^n - 1}{2}\right) \quad \text{since the series is a geometric series} \\
&= 2 + 2(3^n - 1) \\
&= 2 \cdot 3^n.
\end{aligned}$$

This proves that our explicit formula is correct for $n + 1$, and the proof is complete. $\square$