Math3250 Combinatorics Week6 Exam Sample

1 Nine vectors

Let $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$, $\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ $\ldots$, $\begin{pmatrix} a_9 \\ b_9 \\ c_9 \end{pmatrix}$ be nine vectors in $\mathbb{Z}^3$. Prove that at least two of these nine vectors have a sum whose coordinates are all even integers.

2 RIFFRAFF

How many different ways are there to arrange the letters in the word RIFFRAFF? How many different ways are there to arrange the letters in the word RIFFRAFF so that the two R's are not adjacent?

3 Binary words

(i) Let $f(n)$ be the number of binary sequences $a_1, a_2, \ldots, a_n$ (note that this means that each a_i is 0 or 1). Note that $f(0) = 1$ because there is one binary sequence of length 0, empty sequence. Find a simple formula for $f(n)$.

(ii) let $g(n)$ be the number of binary sequences $a_1, a_2, \ldots, a_n$ with no two consecutive 1's. Find a simple formula for $g(n)$. Note that $g(0) = 1$ because there is one binary sequence of length 0, empty sequence. Express your answer in terms of the Fibonacci numbers (given by $F_1 = F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$).

4 Compositions where each part is divisible by three

Let C be the set of compositions of 24 (into any number of parts) such that each part is divisible by 3. How many elements does C have?

5 Bijections

Let $n \geq 4$. How many bijections $\pi : [n] \to [n]$ satisfy $\pi(1) = 2$, $\pi(2) \neq 3$, $\pi(2) \neq 4$, and $\pi(3) \neq 4$? Give a simple formula not involving summation symbols.

(Afterwards, you should check that your formula works for $n = 4$).

6 Finding an identity

Find a simple formula (no summation symbols) for

$$f(n) = \sum_{k=0}^{n} \binom{k}{2} \binom{n}{k}.$$

7 Integer partitions with no parts equal to 1

For $n \geq 2$, let $f(n)$ be the number of (integer) partitions of n with no parts equal to 1. For example, $f(1) = 0$, $f(2) = 1$ because the only such partition is (2), $f(3) = 1$ counts the partition (3), $f(4) = 2$ counts the partitions (4) and $(2, 2)$ $f(5) = 2$ counts the partitions (5) and $(3, 2)$.

Express $f(n)$ in terms of the partition function, i.e. in terms of the numbers $p(1), p(2), p(3), \ldots, p(k)$ where $p(k)$ is the number of partitions of k. Your formula should be simple, containing no summation symbols.

8 Enumerating all subsets

Given a positive integer n, what is the the number of all subsets of $[n]$?

a. Prove by induction on n.

b. Prove by another method.

9 A sequence

Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and let

$$a_{n+1} = 2(a_0 + a_1 + \cdots + a_n)$$

for $n \geq 0$. Prove that $a_n = 2 \cdot 3^{n-1}$ for $n \geq 1$.