MATH3250 COMBINATORICS WEEK5 PROBLEM SET

THE NAMES OF TEAM MEMBERS, IN ALPHABETICAL ORDER

Credit:

Instruction. Create just one Overleaf project where all team members have edit permission. Complete all problems. To submit, please send me an invite via Overleaf.

1. An identity (pick one)

Complete **one** of the following problems.

a.) Let $n \geq 2$ be an integer. Show that

$$\sum_{j=1}^{n} j^2 \binom{n}{j} = n(n+1)2^{n-2}.$$

See hints:

b.) Let k and n be positive integers such that $k < n$. Show that

$$\sum_{j=k}^{n} \binom{j}{k}\binom{n}{j} = \binom{n}{k} 2^{n-k}.$$

See hints:

2. Integrate

Find an explicit (closed-form expression) formula for the expression

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} 5^{k+1}.$$

See hints:

3. ABC Identity

Prove the identity

$$\sum_{a+b+c=n} a \binom{n}{a, b, c} = n\ 3^{n-1}$$

where the sum is taken over all triples (a, b, c) of nonnegative integers satisfying $a + b + c = n$. See hints:

4. Expansion

Compute the expansion of $(1-x)^{-2}$ by hand. (You don't need to show your work.) See hints:

5. Schedule

Suppose you have to do exactly twelve work shifts at the bookstore (which is only open Monday through Friday) each week. Due to your other commitments, you must work at least 3 shifts on Monday, at least 2 shifts on Wednesday, and at least 1 shift on Friday.

a. In how many ways can you do this?
b. Suppose you have one more restriction. Because you are very busy on Tuesday, you can work at most one shift on Tuesday. In how many ways can you arrange your work schedule?

See hints:

6. Parts larger than 2 (find a recurrence relation)

Let $A_0 = 1$, and $A_1 = A_2 = 0$. For $n \geq 3$, let A_n be the number of compositions of n such that each part is larger than 2, that is, compositions $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$ and $a_i \geq 3$ for all i. (For example, $A_3 = A_4 = A_5 = 1$, $A_6 = 2$, $A_7 = 3$, and $A_8 = 4$, and $A_9 = 6$.)

Guess a recurrence relation expressing A_n using earlier terms in this sequence, and prove the recurrence relation.
See hints:

7. Sum of the first parts (find a closed-form expression)

For a positive integer n, let S_n be the sum of the first part of all compositions of n. For example, $S_1 = 1$, $S_2 = 3$, $S_3 = 7$, and $S_4 = 15$. Find and prove an explicit (closed-form expression) formula for S_n.

See hints:

8. Write Your Own Problem

Please write a problem with reasonable difficulty and solve it using theorems or concepts from Chapter 4 or Section 5.1 of Bona. Students' problems may be chosen for future exams' questions.

9. Miscellaneous

i. Briefly summarize your team work process and experience.
ii. Approximately how much time did you spend on this homework?