MATH3250 COMBINATORICS WEEK4 PROBLEM SET

YOUR PREFERRED FIRST AND LAST NAME

Credit: Write down everyone who helped you, including classmates who contributed to your thought process. Write down Bona's textbook and other written sources you used as well.

Instruction. The exact problems will be announced on Monday during class. Please send me an invite via Overleaf. If you are not sure how to do something, please post on Piazza or come to office hour.

For practice, when you work on Problems 1 - 5 below, please model your explanations after the the proofs for Theorem 3.6, Examples 3.7, 3.11, 3.12 in Sec 3.2; Theorems 3.16 in Sec 3.3.

1. Functions

Let n be a positive integer.

(1) How many injective functions are there from $[n]$ to $[n]$? Uncomment for a hint:
(2) How many functions are there from $[n]$ to $[n]$ that are not injective? Uncomment for a hint:

2. Interns

A local company has 8 interns from UConn and 12 interns from Eastern Connecticut State University (ECSU). The company would like to form a service committee consisting of the interns.

(1) How many ways are there to form this committee consisting of 2 UConn interns and 3 ECSU interns?
(2) How many ways are there to form a 5-people committee that contains *at least* one UConn student and one ECSU student?

3. Polygon diagonals

Let $n \geq 4$. Consider a convex n-gon that is drawn in such a way that no three diagonals intersect in one point. How many intersection points do the diagonals have? (For example, if you draw a pentagon, there are five diagonals and five crossings.) Prove this.
See hints by removing the percent sign:

4. Rooks

We would like to place n rooks on an $n \times n$ chess board in such a way that no rook can attack. In chess, a rook is only allowed to move horizontally or vertically, through any number of unoccupied squares. Please see https://en.wikipedia.org/wiki/Rook_(chess) for a demonstration. In how many ways can we do this?

5. Combinatorics class

A Combinatorics class consists of n sophomores, n juniors, and n seniors. For example, our class has 12 students. Pretend that we have exactly 4 sophomores, 4 juniors, and 4 seniors.

(1) The students are to form n presentation groups of three people each. How many ways can they do this if each group must contain a sophomore, a junior and a senior?
(2) The n seniors are to form a circle. Two circle arrangements are considered identical if each person has the same left neighbor in the circles. How many ways can the n seniors to form a circle?

For practice, when you work on Problems 6 - 10 below, please model your explanations after the the proofs for Theorems 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7 from Section 4.1.

6. Sums

Let n be a positive integer. Prove that the identities

$$\sum_{k=0}^{n} 2^k \binom{n}{k} (-1)^{n-k} = 1 \qquad \text{and} \qquad \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

hold.
See hints by removing the percent sign:

7. Inequality

Let n be a positive integer larger than 1. Prove that

$$2^n < \binom{2n}{n} < 4^n$$

Additional problem (Optional): Let k be an integer where $2 \leq k < n$. Show that the inequality

$$k^n < \binom{kn}{n}$$

holds.
See hints by removing the percent sign:

8. An identity - please omit this problem

Let $n \geq 2$ be an integer. Show that
$$\sum_{j=1}^{n} j^2 \binom{n}{j} = n(n+1)2^{n-2}.$$
See hints by removing the percent sign:

9. A more complicated identity - please omit this problem

Let k and n be positive integers such that $k < n$. Show that
$$\sum_{j=k}^{n} \binom{j}{k}\binom{n}{j} = \binom{n}{k} 2^{n-k}.$$
See hints by removing the percent sign:

10. kMn

Let k, M, n be nonnegative integers such that $k + M \leq n$. Prove that
$$\binom{n}{M}\binom{n-M}{k} = \binom{n}{k}\binom{n-k}{M}$$

11. Write Your Own Problem

Please write your own problem and solve it using theorems or concepts from Chapter 3 or Section 4.1 of Bona. Student's problems may be chosen for future exams' questions. Test your problem by sharing it with another person who likes math.

12. Miscellaneous

i. Share your work (at least one problem) and thought process with at least one classmate. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. A virtual discussion via Piazza or email is fine if you don't have time to interact in person.
ii. Approximately how much time did you spend on this homework?