MATH3250 COMBINATORICS WEEK3 PROBLEM SET

YOUR PREFERRED FIRST AND LAST NAME

Credit: Write down everyone who helped you, including classmates who contributed to your thought process (either through sharing insights or through being a sounding board). Write down Bona's textbook and other written sources you used as well.

Instruction. The exact problems will be announced on Monday during class. Please send me an invite via Overleaf.

Note: For your convenience, please use the SAMPLE SOLUTION below as a template for typing up an induction proof.

If you are not sure how to do something, please post on Piazza or come to office hour.

Sample Solution

Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and let

$$a_{n+1} = 2(a_0 + a_1 + \cdots + a_n)$$

for $n \geq 0$. Compute the first few values of a_n, then conjecture an explicit formula for a_n, and then prove the formula using induction.

Solution and proof. We claim that $a_n = 2 \cdot 3^{n-1}$ for $n \geq 1$. We prove this by strong induction on n. Since $2(a_0) = 2(1) = 2 \cdot 3^{1-1}$, the initial case (for $n = 1$) is verified. Now let us assume that the statement is true for all positive integers that are less than or equal to n. Then, we have

$$\begin{aligned}
a_{n+1} &= 2(a_0 + a_1 + a_2 + \cdots + a_n) \quad \text{by the recurrence relation} \\
&= 2a_0 + 2(a_1 + a_2 + \cdots + a_n) \\
&= 2 + 2(2 \cdot 1 + 2 \cdot 3 + \cdots + 2 \cdot 3^{n-1}) \quad \text{by the induction hypothesis} \\
&= 2 + 4(1 + 3 + \cdots + 3^{n-1}) \\
&= 2 + 4\left(\frac{3^n - 1}{2}\right) \quad \text{since the series is a geometric series} \\
&= 2 + 2(3^n - 1) \\
&= 2 \cdot 3^n.
\end{aligned}$$

This proves that our explicit formula is correct for $n+1$, and the proof is complete. □

1. Recurrence relation

Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and

$$a_n = 3(a_0 + a_1 + \cdots + a_{n-1}) + 1$$

for $n > 0$. Compute the first few values of a_n, then conjecture an explicit formula for a_n, and then prove the formula using induction.

Proof. Insert proof □

2. polygon

Prove (using induction) that the sum of the angles of a convex n-gon is $(n-2)180$ degrees.

Note: You may use the fact (possibly proven in your geometry class) about the sum of angles of a convex triangle without proof.

Proof. Insert proof □

3. Divisible by eleven

Prove that a positive integer with digits $a_1, a_2, \ldots, a_n$ is divisible by 11 if and only if $a_1 - a_2 + a_3 - \cdots + (-1)^{n-1}a_n$ is divisible by 11

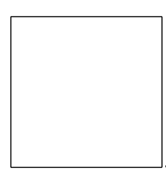
Proof. Insert proof □

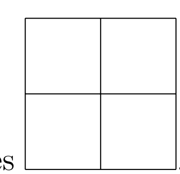
4. Divisible by three

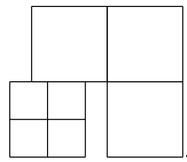
Prove that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Proof. Insert proof □

5. Squares

We start with one square piece of paper .

We then cut this square paper into four smaller squares .

Then cut one of the obtained small squares into four smaller squares, so that we get 7 squares , and so on.

Let a_n be the number of squares we have after the n-th time that we perform this cutting operation. Compute the first few values of a_n (beyond $n = 1$ and $n = 2$ given above), then conjecture an explicit formula for a_n, and then prove the formula using induction.

Proof. Insert proof □

6. 6 digit numbers

a) How many 6 digit numbers are there (leading zeros, e.g. 001223 not allowed)?
b) How many of these are even?
c) How many 6 digit numbers are there with exactly one 7?
d) How many 6 digit numbers are there that are the same forward and backwards (e.g., 890098)?

Proof. Insert answers and explanations □

7. BAA, ABA, AAB

How many 3 digit positive integers contain two (but not three) digits?

8. Sandwich shop

A sandwich shop has 4 protein options. It also has 6 veggies: lettuce, sprouts, carrots, onion, tomato and pickles. It carries 5 sauces: mustard, catsup, mayo, sirachi, and vinegar. How many sandwiches can be made from one protein, one veggie and AT MOST one sauce?

Proof. Insert answer and a brief reasoning. □

9. Connecticut

Compute the number of ways to create a list (of size 11) of the letters of the word CONNECTICUT.

Proof. Insert answer and a brief reasoning. □

10. Alternating parity

a) Warm-up: In how many ways can the elements of [3] be permuted so that the sum of every two consecutive elements in the permutation is odd? In how many ways can the elements of [4] be permuted so that the sum of every two consecutive elements in the permutation is odd?
b) (Optional) Compute this for [5] as well.
c) In how many ways can the elements of $[n]$ be permuted so that the sum of every two consecutive elements in the permutation is odd?

11. Write Your Own Problem

Please write your own problem and solve it using theorems or concepts from Section 2.1-2.2 or Section 3.1 of Bona. (Student's problems may be chosen for future exams' questions). You may test the difficulty level of your problem by sharing it with a classmate.

12. Miscellaneous

i. Share your work (at least one problem) and thought process with at least one classmate. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. A virtual discussion via Piazza or email is fine if you don't have time to interact in person.
ii. Approximately how much time did you spend on this homework?