Math3250 Combinatorics Week15 Final Sample

1 Lattices Sec 16.3

Given one of the following posets P, find the maximum and minimum elements (if possible) and sketch the Hasse diagram. Given two elements in P, find the meet and join (if they exist). Be able to informally reproduce the explanation from the given source.

a. From Example 16.17, Figure 16.5

b. The poset of all positive integers ordered as usual (from Example 16.18).

c. The poset of all positive integers with partial order in which $x \leq y$ iff x is a divisor of y (from Example 16.20).

d. The Boolean algebra (from Example 16.19).

e. The poset of set partitions on [3] or [4] (from Example 16.34).

f. The poset of noncrossing partitions [3] or [4] (from Example 16.36).

g. The poset of integer partitions of 5, 6, or 7 (from `egunawan.github.io/combinatorics/notes/week13_sec16.3lattices.pdf`).

2 Möbius function Sec 16.2 and Sec 16.3

Use Theorem 16.15 or Corollary 16.16 to compute the the values of μ on the intervals of the following poset:

a. From Example 16.17, Figure 16.5

b. The poset of all positive integers ordered as usual (from Example 16.18).

c. The poset of all positive integers with partial order in which $x \leq y$ iff x is a divisor of y (from Example 16.20).

d. The Boolean algebra (from Example 16.19).

e. The poset of set partitions on [3] or [4] (from Example 16.34).

3 Twelvefold way

Compute the answer to each enumeration problem below. Express your answer both in a way that has mathematical meaning (e.g., a difference of binomial coefficients) and as a nonnegative integer. If a problem is a direct consequence of an entry in the twelvefold way, explain which one (e.g., "this is equivalent to putting indistinct balls in distinct boxes surjectively").

a. How many subsets of the set $[12] = \{1, 2, \ldots, 12\}$ contain at least one odd integer?

b. In how many ways can eight people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (not necessarily on the same side)?

c. How many permutations $\pi : [6] \to [6]$ satisfy $\pi(1) \neq \pi(2)$?

d. There are four job openings and six candidates. Each job opening is filled by one of the candidates. In how many ways can this be done?

e. Ten people split up into five groups of two each. In how many ways can this be done?

f. How many compositions of 20 use only the parts 2 and 3?

g. How many partitions of 8 are there into odd parts?

h. In how many different ways can the letters of the word BOOKKEEPER be arranged if the three E?s cannot appear consecutively?

i. How many sequences $(a_1, a_2, \ldots, a_{12})$ are there consisting of four 0?s and eight 1's if no two consecutive terms are both 0's?

4 Weak compositions and OGF

a. What does it mean for a sequence $(a_1, a_2, \ldots, a_k)$ to be a *weak composition* of n? [1].

b. What is the number of weak compositions of n into k parts? [2]

c. Let $b_0 = 1$, and, if $n > 0$, let b_n be the number of weak compositions of n into 5 parts. Let $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be the ordinary generating function of b_n. Give an explicit formula for $B(x)$ as a function of x.

5 Binomial Theorem and OGF

a. Prove that the ordinary generating function for the sequence $c_n = \binom{2n}{n}$ is $(1-4x)^{-\frac{1}{2}}$.

b. Prove that

$$\sum_{i=0}^{n} \binom{2i}{i} \binom{2(n-i)}{n-i} = 4^n.$$

6 Nine vectors

Let $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \ldots, \begin{pmatrix} a_9 \\ b_9 \\ c_9 \end{pmatrix}$ be nine vectors in $\mathbb{Z}^3$. Prove that at least two of these nine vectors have a sum whose coordinates are all even integers.

7 RIFFRAFF

How many different ways are there to arrange the letters in the word RIFFRAFF? How many different ways are there to arrange the letters in the word RIFFRAFF so that the two R's are not adjacent?

8 Binary words

(i) Let $f(n)$ be the number of binary sequences $a_1, a_2, \ldots, a_n$ (note that this means that each a_i is 0 or 1). Note that $f(0) = 1$ because there is one binary sequence of length 0, empty sequence. Find a simple formula for $f(n)$.

(ii) let $g(n)$ be the number of binary sequences $a_1, a_2, \ldots, a_n$ with no two consecutive 1's. Find a simple formula for $g(n)$. Note that $g(0) = 1$ because there is one binary sequence of length 0, empty sequence. Express your answer in terms of the Fibonacci numbers (given by $F_1 = F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$).

9 Compositions where each part is divisible by three

Let C be the set of compositions of 24 (into any number of parts) such that each part is divisible by 3. How many elements does C have?

10 Bijections

Let $n \geq 4$. How many bijections $\pi : [n] \to [n]$ satisfy $\pi(1) = 2$, $\pi(2) \neq 3$, $\pi(2) \neq 4$, and $\pi(3) \neq 4$? Give a simple formula not involving summation symbols.

(Afterwards, you should check that your formula works for $n = 4$).

11 Finding an identity

Find a simple formula (no summation symbols) for

$$f(n) = \sum_{k=0}^{n} \binom{k}{2} \binom{n}{k}.$$

[1] The first definition in Section 5.1

[2] This number is given in the first theorem in Section 5.1.

12 Integer partitions with no parts equal to 1

For $n \geq 2$, let $f(n)$ be the number of (integer) partitions of n with no parts equal to 1. For example, $f(1) = 0$, $f(2) = 1$ because the only such partition is (2), $f(3) = 1$ counts the partition (3), $f(4) = 2$ counts the partitions (4) and $(2, 2)$ $f(5) = 2$ counts the partitions (5) and $(3, 2)$.

Express $f(n)$ in terms of the partition function, i.e. in terms of the numbers $p(1), p(2), p(3), \ldots, p(k)$ where $p(k)$ is the number of partitions of k. Your formula should be simple, containing no summation symbols.

13 Enumerating all subsets

Given a positive integer n, what is the the number of all subsets of $[n]$?

a. Prove by induction on n.

b. Prove by another method.

14 A sequence

Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 1$, and let

$$a_{n+1} = 2(a_0 + a_1 + \cdots + a_n)$$

for $n \geq 0$. Prove that $a_n = 2 \cdot 3^{n-1}$ for $n \geq 1$.