1. Let $a_0 = 1, a_1 = 4$ and $a_{n+2} = 8a_{n+1} - 16a_n$ if $n \geq 0$.

   (a) Use the recurrence relation to find an explicit formula for the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$.

Solution: From Supplementary Chapter 8 Exercise 26. Good for exam because no partial fraction computation is required. (If I want to change the numbers to p, q, r, I can follow Exercise 28.)

Below is copied from Bona 4th ed. Let $A(x) = \sum_{k\geq 0} a_n x^n$. Multiply both sides of the defining recurrence relation by x^{n+2} and sum over $k \geq 0$ to get

$$A(x) - 4x - 1 = 8x(A(x) - 1) - 16x^2 A(x),$$
$$A(x) = \frac{1-4x}{1-8x+16x^2} = \frac{-1+4x}{-1+8x-16x^2} = \frac{1}{1-4x}.$$

   (b) Use the previous part to compute an explicit formula for a_n for $n \geq 0$.

Solution: Therefore, $\boxed{a_n = 4^n}$ for $n \geq 2$.

2. For $n \geq 0$, let $p(n)$ denote the number of partitions of the integer n. Recall the fact (which you don't need to prove) that $\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$.

   Recall also that the ordinary generating function for the number of partitions of n for which no part is divisible by 3 is equal to the number of partitions of n for which no part appears more than twice, $\prod_{i\geq 1} \frac{1-x^{3i}}{1-x^i} = \prod_{i\geq 1}(1+x^i+x^{i2})$.

   (a) Let $a_0 = 1$, and let a_n be the number of partitions of n for which no part is divisible by 3 *and* no part appears more than twice.

      i. Write down all such partitions for $n = 5$

      Solution: There are three

      ii. Write down a formula (not as an infinite series) for the ordinary generating function for a_n. Briefly justify your formula (but you don't need to write a complete proof).

      Solution:

      $$\prod_{i \text{ does not divide } 3} (1+x^i+x^{i2})$$

      iii. Does the coefficient for x^5 for your generating function matches the number of partitions you wrote down in the first part?

(b) Let $b_0 = 1$ and let b_n be the number of partitions of n for which no part is bigger than 3.

i. Write down all such partitions for $n = 5$

Solution: There are five: (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).

ii. Write down a formula (not as an infinite series) for the ordinary generating function for b_n. Briefly justify your formula (but you don't need to write a complete proof).

Solution:

$$\prod_{i\geq 1} \frac{1}{(1-x)(1-x^2)(1-x^3)}$$

iii. Does the coefficient for x^5 for your generating function matches the number of partitions you wrote down in the first part?

(c) Is a_n and b_n the same sequence? Prove your answer.

3. (a) Let $a_0 = 1$, $a_1 = 1$, and let

$$a_n = n\ a_{n-1} + n\ (n-1)\ a_{n-2} \quad \text{for } n \geq 2.$$

Find an explicit formula (as a function of x, not involving an infinite sum) for the exponential generating function $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

Solution:

$$A - x - 1 = x(A-1) + x^2 A$$

$$\boxed{A(x) = \frac{1}{1-x-x^2}}$$

(b) You have seen this function before as an *ordinary* generating function for a different sequence $\{b_n\}_{n=0}^{\infty}$. Give an explicit formula for b_n.

Solution:
See the "child walks up a stairway" problem in chapter 8 and the answer key in the back of the chapter.

(c) Give a recurrence relation for the mystery sequence b_n.

Solution: $b_n = b_{n-1} + b_{n-2}$

4. Recall that $\binom{m}{0} := 1$, $\binom{m}{1} = m$, and $\displaystyle\binom{m}{k} := \frac{m(m-1)\dots(m-k+1)}{k!}$.

(a) Prove that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0} \binom{2n}{n} x^n.$$

Solution: From Exercise 27 Bona 4th ed (answer and computation is in book).

(b) Compute the power series of $(1-x)^{-5}$ using the Binomial Theorem.

(c) Let $a_0 = 1$ and let a_n be the number of weak compositions into 5 parts. Recall that the definition of weak compositions and a formula are in the first two pages of Section 5.1. Memorize or practice writing the proof for the formula.

(d) Give an explicit formula (as a function of x, without an infinite sum) of the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$. [1] Justify your answer.

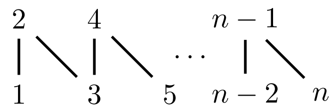
Solution: The answer is $\boxed{(1-x)^{-5}}$ because the coefficient for x^n from part (b) is equal to the formula for a_n in part (c).

(e) For a positive integer m, compute the power series of $(1-x)^{-m}$ using the Binomial Theorem. What is the ordinary generating function for the number of weak compositions into m parts?

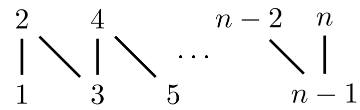
[1] Hint: Look at part (b).

5. Submit the optional Problem 3 from Week 13 Problems.

Let a_n be the number of antichains of the poset P_n whose Hasse diagram is



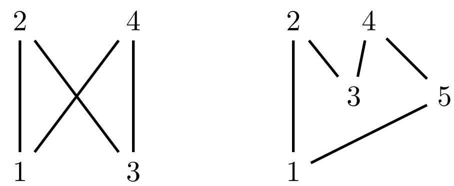
if n is odd and



if n is even.

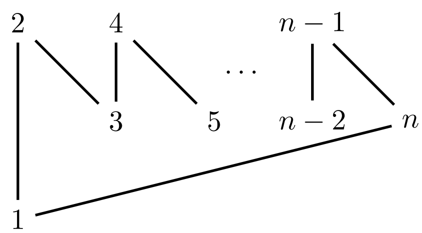
The recurrence relation for a_n is given in the answer key of the previous problem set.

Let P'_4 and P'_5 be posets whose Hasse diagrams are shown below.

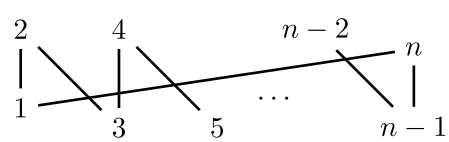


Let J_4 (resp. J_5) be the set of all order filters of the poset P'_4 (resp. P'_5), and define a partial order by inclusion, that is, $F_1 \leq F_2$ iff $F_1 \subset F_2$. Recall that there are 7 and 10 order filters (respectively) in J_4 and J_5, including the empty set.

For $n \geq 4$, let b_n be the number of the order filters of the n-element poset P'_n whose Hasse diagram is



if n is odd and



if n is even.

a.) **Prove** that, if n is even and at least 4, then

$$b_n = a_{n-3} + a_{n-1} \tag{1}$$

Solution:

We have proven that the number of antichains of a poset is equal to the number of order filters of the same poset, hence b_n is the number of antichains of P'_n. To prove

$$b_n = 2a_{n-3} + a_{n-2} \text{ for all even } n \geq 4,$$

suppose that $n \geq 4$ is even, and let A be an antichain of P'_n.
First, suppose $n \in A$. Then $1 \notin A$ and $n-1 \notin A$ because both 1 and $n-1$ are comparable to n. However, none of the numbers $2, \ldots, n-2$ is comparable to n, and there are a_{n-3} ways to pick an antichain of the subposet $\{2, \ldots, n-2\}$ of P_n.
Next, suppose $n \notin A$. If $1 \in A$, then $2 \notin A$ and $n \notin A$ because both 2 and n are comparable to 1. However, none of the numbers $3, \ldots, n-1$ are comparable to 1, and the number of options is equivalent to counting the number of antichains of the poset P_{n-3}, which is a_{n-3}.
If $1, n \notin A$, then the elements that may go in A are $2, 3, \ldots, n-1$, so the number of options is the same as counting the number of antichains of a poset isomorphic to the poset P_{n-2}, which is a_{n-2}.

b.) **Prove** that, if n is even, then b_n is a Lucas number. [2]

Solution: Let ℓ_n be defined by $\ell_2 = 3$, $\ell_3 = 4$, and $\ell_n = 2a_{n-3} + a_{n-2}$ for $n \geq 4$. Then

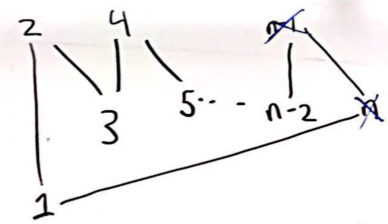
$$\begin{aligned}
\ell_{n+1} + \ell_n &= (2a_{n-2} + a_{n-1}) + (2a_{n-3} + a_{n-2}) \\
&= a_{n-2} + a_{n-2} + a_{n-1} + a_{n-3} + a_{n-3} + a_{n-2} \\
&= a_{n-2} + a_n + a_{n-3} + a_{n-1} \quad \text{since } a_n \text{ are the Fibonacci numbers} \\
&= a_{n-1} + a_{n-1} + a_n \\
&= 2a_{n-1} + a_n \\
&= \ell_{n+2}.
\end{aligned}$$

Thus each ℓ_n is a Lucas number. Since $b_n = \ell_n$ whenever n is even, the statement follows.

c.) **Prove** that, if n is odd, then $b_n = 2\ a_{n-2}$.

Solution:

[2] Hint: Use equation (1) and apply arithmetic.

$P =$ 

$a_n :=$ antichains of P

$b_n :=$ order filters of P

n odd

$a_n = a_{n-1} + a_{n-2}$

Since there is a bijection from $\{\text{order filters of } P\} \to \{\text{antichains of } P\}$, we can consider antichains of P.

There are two possible cases for any antichain of P:

Case 1: n is in the antichain

Any antichain of P that includes n cannot also include $n-1$ or 1, because n is comparable to both 1 and $n-1$. Thus, we form an antichain from a poset with the structure above with $n-3$ elements, of which there are exactly a_{n-3} antichains.

Case 2: n is not in the antichain

If n is not in the antichain, then we consider antichains formed from the poset with the same structure as P with $n-1$ elements.

We have two possible sub-cases:

- Case A: 1 is in the antichain

If 1 is in the antichain, we can't have 2 or $n-1$ in the antichain because 1 is comparable to both 2 and $n-1$. Thus, we form an antichain from a poset with the structure above with $n-4$ elements, of which there are exactly a_{n-4} antichains.

- Case B: 1 is not in the antichain

If 1 is not in the antichain, then we form an antichain from a poset with the same structure above with $n-2$ elements of which there are exactly a_{n-2} antichains.

Combining all the cases, we have $b_n = a_{n-2} + \underbrace{a_{n-3} + a_{n-4}}_{a_{n-2}} = 2a_{n-2}$. □

Extra credit

Write a problem using techniques and concepts related to set partition, poset or generating function. Write a brief, correct solution key.

Extra credit

Describe an interesting theorem or idea from this class since the first test.