1. Let $a_0 = 1, a_1 = 4$ and $a_{n+2} = 8a_{n+1} - 16a_n$ if $n \geq 0$.

   (a) Use the recurrence relation to find an explicit formula for the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$.

   (b) Use the previous part to compute an explicit formula for a_n for $n \geq 0$.

2. For $n \geq 0$, let $p(n)$ denote the number of partitions of the integer n. Recall the fact (which you don't need to prove) that $\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$.

   Recall also that the ordinary generating function for the number of partitions of n for which no part is divisible by 3 is equal to the number of partitions of n for which no part appears more than twice, $\prod_{i\geq 1} \frac{1-x^{3i}}{1-x^i} = \prod_{i\geq 1}(1+x^i+x^{i2})$.

   (a) Let $a_0 = 1$, and let a_n be the number of partitions of n for which no part is divisible by 3 *and* no part appears more than twice.

      i. Write down all such partitions for $n = 5$

      ii. Write down a formula (not as an infinite series) for the ordinary generating function for a_n. Briefly justify your formula (but you don't need to write a complete proof).

      iii. Does the coefficient for x^5 for your generating function matches the number of partitions you wrote down in the first part?

(b) Let $b_0 = 1$ and let b_n be the number of partitions of n for which no part is bigger than 3.

i. Write down all such partitions for $n = 5$

ii. Write down a formula (not as an infinite series) for the ordinary generating function for b_n. Briefly justify your formula (but you don't need to write a complete proof).

iii. Does the coefficient for x^5 for your generating function matches the number of partitions you wrote down in the first part?

(c) Is a_n and b_n the same sequence? Prove your answer.

3. (a) Let $a_0 = 1$, $a_1 = 1$, and let

$$a_n = n\, a_{n-1} + n\,(n-1)\, a_{n-2} \quad \text{for } n \geq 2.$$

Find an explicit formula (as a function of x, not involving an infinite sum) for the exponential generating function $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

(b) You have seen this function before as an *ordinary* generating function for a different sequence $\{b_n\}_{n=0}^{\infty}$. Give an explicit formula for b_n.

(c) Give a recurrence relation for the mystery sequence b_n.

4. Recall that $\binom{m}{0} := 1$, $\binom{m}{1} = m$, and $\displaystyle\binom{m}{k} := \frac{m(m-1)\dots(m-k+1)}{k!}$.

   (a) Prove that
   $$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0} \binom{2n}{n} x^n.$$

   (b) Compute the power series of $(1-x)^{-5}$ using the Binomial Theorem.

   (c) Let $a_0 = 1$ and let a_n be the number of weak compositions into 5 parts. Recall that the definition of weak compositions and a formula are in the first two pages of Section 5.1. Memorize or practice writing the proof for the formula.

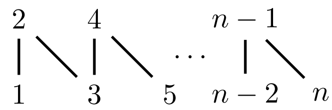
   (d) Give an explicit formula (as a function of x, without an infinite sum) of the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$. [1] Justify your answer.

   (e) For a positive integer m, compute the power series of $(1-x)^{-m}$ using the Binomial Theorem. What is the ordinary generating function for the number of weak compositions into m parts?

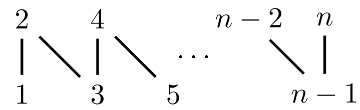
[1] Hint: Look at part (b).

5. Submit the optional Problem 3 from Week 13 Problems.

   Let a_n be the number of antichains of the poset P_n whose Hasse diagram is



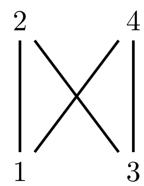
if n is odd and

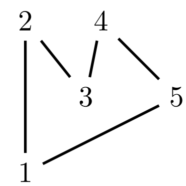


if n is even.

   The recurrence relation for a_n is given in the answer key of the previous problem set.

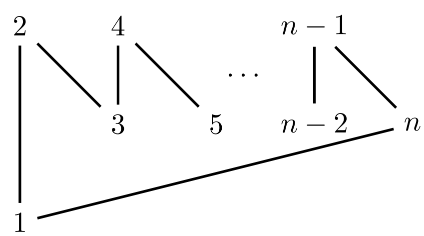
   Let P'_4 and P'_5 be posets whose Hasse diagrams are shown below.



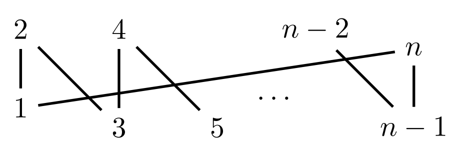


   Let J_4 (resp. J_5) be the set of all order filters of the poset P'_4 (resp. P'_5), and define a partial order by inclusion, that is, $F_1 \leq F_2$ iff $F_1 \subset F_2$. Recall that there are 7 and 10 order filters (respectively) in J_4 and J_5, including the empty set.

   For $n \geq 4$, let b_n be the number of the order filters of the n-element poset P'_n whose Hasse diagram is



if n is odd and



if n is even.

   a.) **Prove** that, if n is even and at least 4, then

$$b_n = a_{n-3} + a_{n-1} \tag{1}$$

   b.) **Prove** that, if n is even, then b_n is a Lucas number. [2]

   c.) **Prove** that, if n is odd, then $b_n = 2\ a_{n-2}$.

[2] Hint: Use equation (1) and apply arithmetic.

Extra credit

Write a problem using techniques and concepts related to set partition, poset or generating function. Write a brief, correct solution key.

Extra credit

Describe an interesting theorem or idea from this class since the first test.