MATH 3250 Combinatorics Week 13 Problem Set

1 Lattice

a. i. Write down and memorize the definition of a lattice. [1]

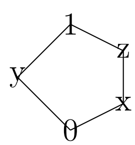
ii. An example of a poset that is not a lattice is given in Sec 16.3. Find a different (connected) poset from this homework or from the book (from Sec 16.1-16.2 only) which is *not* a lattice, and explain why it fails to be a lattice.

iii. Describe or draw the Hasse diagram of a poset which *is* a lattice. Choose a poset which is not a chain and has more than 5 elements. [2]

b. i. Give a counterexample to the following statement[3]:

$$\text{If } L \text{ is a lattice and } x \leq z, \text{ then } x \vee (y \wedge z) = (x \vee y) \wedge z \text{ for all } y \in L.$$

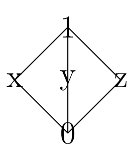
Solution: A poset having this property is called *modular*. See Problem 11 from Bona (3rd and 4th edition).

A counterexample is 

. For example, verify that $x \vee (y \wedge z) \neq (x \vee y) \wedge z$.

c. i. Give a counterexample to the following statement[4]:

$$\text{If } L \text{ is a lattice, then } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ for all } x, y, z \in L.$$

Solution: A lattice having this property is called *distributive*. A counterexample is the poset 

. Check that $x \vee (y \wedge z) \neq (x \vee y) \wedge (x \vee z)$.

ii. Give one lattice[5] where $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, z in this lattice.

2 Distributive lattice of order filters

Definition. We say that a lattice L is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$.

Let P be a finite poset. Consider a new poset $J(P) := \{$ order filters of P $\}$ ordered by inclusion, that is, $F_1 \leq F_2$ iff $F_1 \subset F_2$.

[1] Sec 16.3, page 431

[2] Several examples are spelled out in Sec 16.3.

[3] Hint: A smallest counterexample is a 5-element poset. You can find a counterexample in Chapter 16.

[4] Hint: A smallest counterexample is a 5-element poset. You can find a counterexample in Chapter 16.

[5] Look for a small example in Chapter 16

a.) **Prove** that $J(P)$ is a lattice.

b.) **Prove** that $J(P)$ is a *distributive* lattice.

Solution:

(a.) To prove that $J(P)$ is a lattice it is enough to show that any two elements F_1 and F_2 of $J(P)$ have a minimum common upper bound and a maximum common lower bound.

Let F and F' be two order ideals of P. Then $F \cap F'$ and $F \cup F'$ are also order filters of P. The filter $F \cap F'$ is larger than any other filter contained in both F and F', so it is the maximum common lower bound of F and F'. The filter $F \cup F'$ is smaller than any other filter containing both F and F', so it is the minimum common upper bound of F and F'.

(b.) As explained in the proof of part (a), $F \wedge F' = F \cap F'$ and $F \vee F' = F \cup F'$ for all $F, F' \in J(P)$.

To prove that the lattice $J(P)$ is distributive, let x, y, z be filters of P. (Our goal is to show that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.) Then

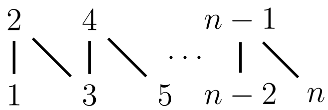
$$\begin{aligned} x \wedge (y \vee z) &= x \cap (y \cup z) \\ &= (x \cap y) \cup (x \cap z) \text{ as proven in your intro proofs class} \\ &= (x \wedge y) \vee (x \wedge z). \end{aligned}$$

c.) Optional: Consider the set of all antichains of P. Define a partial order on this set so that it is a distributive lattice. [6]

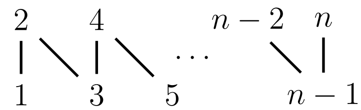
[6] See the previous problem set, where you gave a bijection between antichains and order filters

3 OPTIONAL: Counting filters

Let a_n be the number of antichains of the poset P_n whose Hasse diagram is



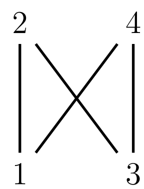
if n is odd and

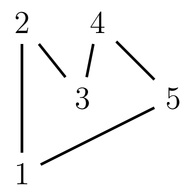


if n is even.

The recurrence relation for a_n is given in the answer key of the previous problem set.

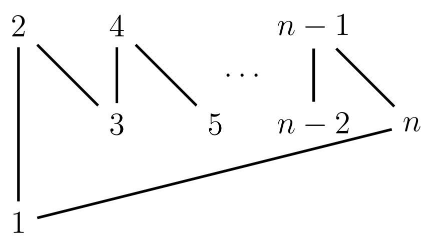
Let P'_4 and P'_5 be posets whose Hasse diagrams are shown below.



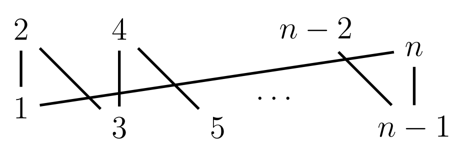


Let J_4 (resp. J_5) be the set of all order filters of the poset P'_4 (resp. P'_5), and define a partial order by inclusion, that is, $F_1 \leq F_2$ iff $F_1 \subset F_2$. Recall that there are 7 and 10 order filters (respectively) in J_4 and J_5, including the empty set.

For $n \geq 4$, let b_n be the number of the order filters of the n-element poset P'_n whose Hasse diagram is



if n is odd and



if n is even.

a.) **Prove** that, if n is even, then

$$b_n = a_{n-3} + a_{n-1} \quad \text{for all } n \geq 4 \, , \tag{1}$$

b.) **Prove** that, if n is even, then b_n is a Lucas number. [7]

c.) **Prove** that, if n is odd, then $b_n = 2\, a_{n-2}$.

[7] Hint: Use equation (1) and apply arithmetic.