<u>**MATH 3250 Combinatorics Week 13 Problem Set**</u>

1 Lattice

a. i. Write down and memorize the definition of a lattice. [1]

ii. An example of a poset that is not a lattice is given in Sec 16.3. Find a different (connected) poset from this homework or from the book (from Sec 16.1-16.2 only) which is *not* a lattice, and explain why it fails to be a lattice.

iii. Describe or draw the Hasse diagram of a poset which *is* a lattice. Choose a poset which is not a chain and has more than 5 elements. [2]

b. i. Give a counterexample to the following statement[3]:

If L is a lattice and $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $y \in L$.

c. i. Give a counterexample to the following statement[4]:

If L is a lattice, then $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$.

ii. Give one lattice[5] where $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, z in this lattice.

2 Distributive lattice of order filters

Definition. We say that a lattice L is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$.

Let P be a finite poset. Consider a new poset $J(P) := \{$ order filters of P $\}$ ordered by inclusion, that is, $F_1 \leq F_2$ iff $F_1 \subset F_2$.

a.) **Prove** that $J(P)$ is a lattice.

b.) **Prove** that $J(P)$ is a *distributive* lattice.

c.) Optional: Consider the set of all antichains of P. Define a partial order on this set so that it is a distributive lattice. [6]

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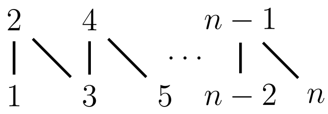
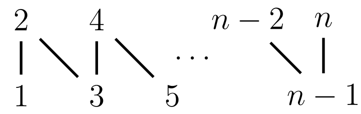
[1] Sec 16.3, page 431
[2] Several examples are spelled out in Sec 16.3.
[3] Hint: A smallest counterexample is a 5-element poset. You can find a counterexample in Chapter 16.
[4] Hint: A smallest counterexample is a 5-element poset. You can find a counterexample in Chapter 16.
[5] Look for a small example in Chapter 16
[6] See the previous problem set, where you gave a bijection between antichains and order filters

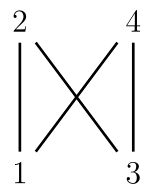
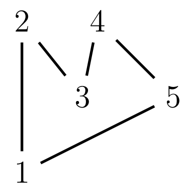
3 OPTIONAL: Counting filters

Let a_n be the number of antichains of the poset P_n whose Hasse diagram is


if n is odd and

if n is even.

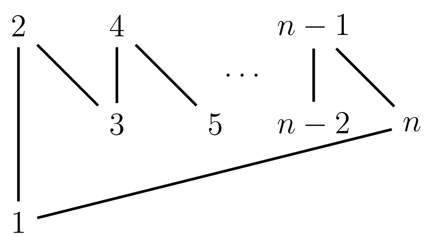
The recurrence relation for a_n is given in the answer key of the previous problem set.

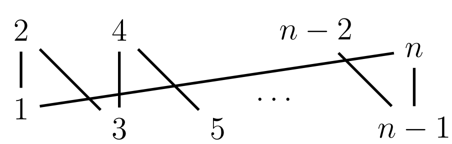
Let P'_4 and P'_5 be posets whose Hasse diagrams are shown below.

Let J_4 (resp. J_5) be the set of all order filters of the poset P'_4 (resp. P'_5), and define a partial order by inclusion, that is, $F_1 \leq F_2$ iff $F_1 \subset F_2$. Recall that there are 7 and 10 order filters (respectively) in J_4 and J_5, including the empty set.

For $n \geq 4$, let b_n be the number of the order filters of the n-element poset P'_n whose Hasse diagram is


if n is odd and


if n is even.

a.) **Prove** that, if n is even, then

$$b_n = a_{n-3} + a_{n-1} \quad \text{for all } n \geq 4 \; , \tag{1}$$

b.) **Prove** that, if n is even, then b_n is a Lucas number. [7]

c.) **Prove** that, if n is odd, then $b_n = 2\, a_{n-2}$.

[7] Hint: Use equation (1) and apply arithmetic.