1 Strong induction with sequences

Let $f_0 := 0$, $f_1 := 1$, and

$$f_n := f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2.$$

Let $\gamma := \frac{1}{2}(1+\sqrt{5})$ and $\delta := \frac{1}{2}(1-\sqrt{5})$. Use strong induction[1] to prove that

$$f_n = \frac{1}{\sqrt{5}}(\gamma^n - \delta^n) \quad \text{for all } n \geq 0. \tag{1}$$

Solution:

Proof. Base step. Equality (1) holds for $n = 0$ since

$$\begin{aligned}\frac{1}{\sqrt{5}}(\gamma^0 - \delta^0) &= \frac{1}{\sqrt{5}}(0)\\ &= 0\\ &= f_0 \quad \text{by definition}\end{aligned}$$

and for $n = 1$ since

$$\begin{aligned}\frac{1}{\sqrt{5}}(\gamma^1 - \delta^1) &= \frac{1}{\sqrt{5}}\left[\frac{1}{2}\left(1+\sqrt{5}\right) - \frac{1}{2}\left(1-\sqrt{5}\right)\right]\\ &= \frac{1}{\sqrt{5}}\left(\sqrt{5}\right)\\ &= 1\\ &= f_1 \quad \text{by definition.}\end{aligned}$$

Inductive step. Let $n \geq 2$, and suppose that equality (1) holds for all nonnegative integers that are less than or equal to n. Then

$$\begin{aligned}f_{n+1} &= f_n + f_{n-1} \quad \text{by definition}\\ &= \frac{1}{\sqrt{5}}(\gamma^n - \delta^n) + \frac{1}{\sqrt{5}}(\gamma^{n-1} - \delta^{n-1}) \quad \text{by the inductive hypohesis}\\ &= \frac{1}{\sqrt{5}}\left[(\gamma^n + \gamma^{n-1}) - (\delta^n + \delta^{n-1})\right]\\ &= \frac{1}{\sqrt{5}}\left[(\gamma^{n-1}(\gamma+1) - \delta^{n-1}(\delta+1)\right]\\ &= \frac{1}{\sqrt{5}}\left[\gamma^{n-1}(\gamma^2) - \delta^{n-1}(\delta^2)\right] \quad \text{because } \gamma+1=\gamma^2 \text{ and } \delta+1=\delta^2\\ &= \frac{1}{\sqrt{5}}\left[\gamma^{n+1} - \delta^{n+2}\right].\end{aligned}$$

□

2 Integer partitions

Let $k, n \geq 1$.

[1] You can read or do on your own a proof using ordinary generating function in the book's solution to Exercise 5. I ask you to prove this statement using strong induction because I want you to get better at induction.

a. Let $p_k(n)$ denote the number of all partitions of (the integer) n into *exactly* k parts. Prove that $p_k(n)$ is equal to the number of partitions of n in which the largest part is exactly k.

Solution: First, note that

$$\{ \text{ (The Ferrers shape of) a partition of } n \text{ into exactly } k \text{ parts } \} =$$
$$\{ \text{ (The Ferrers shape of) a partition of } n \text{ which has exactly } k \text{ rows } \}$$
$$\text{and}$$
$$\{ \text{ (The Ferrers shape of) a partition of } n \text{ in which the largest part is } \textit{exactly } k \ \} =$$
$$\{ \text{ (The Ferrers shape of) a partition of } n \text{ which has exactly } k \text{ columns } \}.$$

One can see that these two sets have the same size by taking conjugates.

b. Prove that the number of partitions of $n + k$ into exactly k parts is equal to the number of partitions of n into at most k parts is equal. That is, let $p_{\leq k}(n)$ denote the number of partitions of n into at most k parts and prove that $p_k(n + k) = p_{\leq k}(n)$.

Solution: First, note that

$$X := \{ \text{ (The Ferrers shape of) a partition of } n+k \text{ into exactly } k \text{ parts } \} =$$
$$\{ \text{ (The Ferrers shape of) a partition of } n+k \text{ which has exactly } k \text{ rows } \}$$
$$\text{and}$$
$$Y := \{ \text{ (The Ferrers shape of) a partition of } n \text{ into at most } k \text{ parts } \} =$$
$$\{ \text{ (The Ferrers shape of) a partition of } n \text{ which has } k \text{ or fewer rows } \}.$$

Take a Ferrers shape in X and remove a box from the end of each of the k nonempty rows. We are left with a Ferrers shape with n boxes in k or fewer rows, which is an element in Y.

To see that this is a bijection, it suffices to show that for all Ferrers shapes f with at most k rows and n boxes, one can find a unique Ferrers shape whose image is f. That shape can be obtained by simply adding an extra box to the end of each of row. If f had fewer than k rows, then add additional rows of length one so that the shape has k rows.

3 OGF for integer partitions

For $n \geq 0$, let $p(n)$ denote the number of partitions of the integer n. Recall that

$$\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}$$

a. Fix a positive integer k. Let $p_k(n)$ denote the number of all partitions of (the integer) n into *exactly* k parts. Find the ordinary generating function of the sequence $p_k(n)$.

Solution: See Sec 8.1.2 (second half).

It follows from Problem 2 that

$$p_k(n) = p_{\leq k}(n - k). \qquad (2)$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} p_k(n)\ x^n &= \sum_{n=k}^{\infty} p_{\le k}(n-k)\ x^n \quad \text{by equation (2)}\\ &= \sum_{n=0}^{\infty} p_{\le k}(n)\ x^{n+k}\\ &= x^k \sum_{n=0}^{\infty} p_{\le k}(n)\ x^n\\ &= x^k(1+x^{1.1}+x^{1.2}+x^{1.3}+\dots)(1+x^{2.1}+x^{2.2}+x^{2.3}+\dots)\dots(1+x^{k.1}+x^{k.2}+x^{k.3}+\dots)\end{aligned}$$

To explain the last equality, replace this sentence with the explanation from Example 8.9.

Hence, the generating function is

$$\boxed{\frac{x^k}{(1-x)(1-x^2)\dots(1-x^k)}}.$$

b. Let $c_0 = 1$ and, if $n \ge 1$, let c_n denote number of partitions of n for which no part appears more than three times. Let $C(x) := \sum_{n=0}^{\infty} c_n x^n$. Find an explicit formula for $C(x)$.

Solution:

$$C(x) = \prod_{i\ge 1}(1+x^i+x^{i2}+x^{i3})$$

Insert here the same (shorter) proof for the formula for $A(x)$ egunawan.github.io/combinatorics/hw/week8key.pdf but replace 2 (a part can appear at most twice) with 3 (a part can appear at most 3 times).

c. Let $d_0 = 1$ and, if $n \ge 1$, let d_n denote number of partitions of n for which no part is divisible by 4. Let $D(x) := \sum_{n=0}^{\infty} d_n x^n$. Find an explicit formula for $D(x)$.

Solution:

$$D(x) = \prod_{i\ge 1 \text{ and } i \text{ not divisible by } 4} \frac{1}{1-x^i} = \prod_{m\ge 0} \frac{1}{1-x^{4m+1}}\frac{1}{1-x^{4m+2}}\frac{1}{1-x^{4m+3}}$$

Insert here the same (shorter) proof for the formula for $B(x)$ egunawan.github.io/combinatorics/hw/week8key.pdf but replace 3 (a part cannot be divisible by 3) with 4 (a part cannot be divisible by 4).

d. Show that the number of partitions of n for which no part appears more than thrice is equal to the number of partitions of n for which no part is divisible by 4.

Solution: Insert here the same proof for the formula for $B(x)$ in Section 4 of egunawan.github.io/combinatorics/hw/week8key.pdf but replace $(1+x^i+x^{i2})(1-x^i) = 1-x^{3i}$ with $(1+x^i+x^{i2}+x^{i3})(1-x^i) = 1-x^{4i}$.

4 OGF Lucas and other Fibonacci-like numbers

Compute the ordinary generating function for sequences $\{a_n\}$ with the same recurrence relation as the Fibonacci number (with various starting values, for example $a_0 = 5, a_1 = 2$ or $a_1 = 3, a_2 = 4$). Watch Lecture video by Jim Fowler for computing a formula for the Fibonacci numbers.

5 OGF Tower of Hanoi

In the "Tower of Hanoi" puzzle, you begin with a pyramid of n disks stacked around a center pole, with the disks arranged from the largest diameter on the bottom to the smallest diameter on top. There are also two empty poles that can accept disks. The object of the puzzle is to move the entire stack of disks to one of the other poles, subject to three constraints:

a. Only one disk may be moved at a time.

b. Disks can be placed only on one of the other three poles.

c. A larger disk cannot be placed on a smaller one.

Let a_n be the number of moves required to move the entire stack of n disks to another pole. Here $a_0 = 0$. Compute the ordinary generating function of a_n.

Solution: Clearly, $a_1 = 1$.

To move n disks, we must first move the $n-1$ top disks to one of the other poles, then move the bottom disk to the third pole, then move the stack of $n-1$ disks to that pole, so we have the recurrence relation

$$a_n = 2a_{n-1} + 1 \text{ for } n \geq 1.$$

Let $A(x)$ be the generating function $\sum_{n=0}^{\infty} a_n x^n$ of a_n. Multiplying the recurrence relation by x^n and summing over all $n \geq 1$, we get

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (2a_{n-1} + 1)x^n, \text{ so}$$

$$\begin{aligned}
A(x) &= a_0 + \sum_{n=1}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=1}^{\infty} (2a_{n-1} + 1)x^n \\
&= a_0 + x \sum_{n=1}^{\infty} (2a_{n-1} + 1)x^{n-1} \\
&= a_0 + x \sum_{n=0}^{\infty} (2a_n + 1)x^n \\
&= a_0 + 2x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} x^n \\
&= a_0 + xA(x) + \frac{x}{1-x}.
\end{aligned}$$

So

$$\boxed{A(x) = \frac{x}{(1-2x)(1-x)} + \frac{a_0}{1-2x}.}$$

6 OGF from recurrence relation

Consider the sequence defined recursively by $r_0 = 3$, $r_1 = 4$, and $r_n = r_{n-1} + 6r_{n-2}$, for $n \geq 2$. Find a closed form expression for the ordinary generating function $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and use this to find a closed form expression for r_n itself.

Solution: Carefully taking into account the two initial terms, the recursive formula for r_n yields

$$R(x) - xR(x) - 6x^2R(x) = 3 + x.$$

Therefore,

$$\boxed{R(x) = \frac{3+x}{1-x-6x^2} = \frac{1}{1+2x} + \frac{2}{1-3x}},$$

where the last equality uses the method of partial fractions. Now expanding these two geometric series yields the explicit formula $\boxed{r_n = (-2)n + 2(3)^n}$ for n ≥ 0.

7 OGF computation using binomial theorem

a. Write down the definition of $\binom{m}{k}$ for any $m \in \mathbb{R}$ and any $k \in \mathbb{Z}_{>0}$. [2]

b. Write $\sqrt{1-4x}$ as a power series $\sum_{n\geq 0} c_n x^n$.

Solution: Explained in Example 4.16 (Section 4.3) of Bona and also in the last page of Section 8.1.2 Catalan notes. The answer is

$$\boxed{\sqrt{1-4x} = 1 - 2x - \frac{2}{n}\sum_{n=2}^{\infty}\binom{2n-2}{n-1}x^n}$$

c. For the exam, practice doing computation using the binomial theorem with a fractional or negative exponent. For more practice, compute the power series form of the expressions in Quick Check Sec 4.3 ($(1-x)^{-m}$ and $\sqrt{1+x}$) and Chapter 4 Exercises 26, 27, 28 ($\sum_{n=1}^{\infty} nx^{n-1}$ and $\frac{1}{\sqrt{1-4x}}$ and $(1-x)(1-x^2)^{1/2}$).

8 OGF and Catalan numbers

Let $b_0 = 1$, and let b_n be the number of triangulations of a regular polygon with $n+2$ vertices for $n \geq 1$. Here are the first few values of b_n.

n	b_n
0	1
1	1
2	2
3	5
4	14
5	42

a. Prove that

$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + b_2 b_{n-3} + \cdots + b_{n-1} b_0.$$

b. Give an explicit formula for the ordinary generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n$ for b_n.

c. Give an explicit formula for b_n.

Solution: Read the solution explained in Section 8.1.2.1 of Bona or copy Section 8.1.2 Catalan notes.

[2] Definition 4.14 in Section 4.3.

9 EGF from recurrence relation

Practice with Example 8.17, 8.19 from Section 8.2 of Bona.

10 EGF the product formula

Let $f(n)$ be the number of ways to do the following. There are n (distinguishable, off course) children in a classroom. You give an odd number of the children either a red candy or a turquoise candy to eat. I give an odd number of the children either a black marble, a purple marble, or a green marble. The remaining children get nothing. (No child receives more than one item.)

For instance, f(1) = 0 (since there must be at least one child that gets a candy and at least one other child that gets a marble) and f(2) = 12 (two choices for which child gets a candy, two choices for the candy color, and three choices for the marble color).

a. Find a simple formula for the exponential generating function $F(x) = \sum_{n=0}^{\infty} f(n)/n!x^n$ not involving any summation symbols.

b. Find a simple formula for f(n) not involving any summation symbols.

Solution: For children who get a candy, the EGF is

$$A(x) = \sum_{n=0}^{\infty} 2^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \frac{1}{2}(e^{2x} - e^{-2x}).$$

The first equality is because the number of ways to give either a red or turquoise candy an odd number k of children is 2^k (for each of the k individuals, either give a red candy or the other color) and the number of ways to give candy to an even number of children is 0.

The second equality is due to

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$
$$e^{-2x} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!}$$
$$\overline{\phantom{e^{2x} - e^{-2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} - \sum_{n=0}^{\infty}}} -$$
$$e^{2x} - e^{-2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!}$$
$$= 2\sum_{n=0}^{\infty} \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}$$

For children who get a marbles, the EGF is

$$B(x) = \sum_{n=0}^{\infty} 3^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \frac{1}{2}(e^{3x} - e^{-3x}).$$

The first equality is because the number of ways to give either a black, purple, or green marble to an odd number k of children is 3^k (for each of the k individuals, there are three colors to choose from) and the number of ways to give marbles to an even number of children is 0.

The second equality is due to

$$\begin{aligned}
e^{3x} &= \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} \\
e^{-3x} &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} \\
\hline
e^{3x} - e^{-3x} &= \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!} \\
&= 2\sum_{n=0}^{\infty} \frac{3^{n+1} x^{2n+1}}{(2n+1)!}
\end{aligned}$$

For the remaining children (who does not get anything), the EGF is

$$C(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n)\frac{x^n}{n!} &= A(x)\ B(x)\ C(x) \\
&= \frac{1}{2}(e^{2x} - e^{-2x})\frac{1}{2}(e^{3x} - e^{-3x})e^x \\
&= \boxed{\frac{1}{4}(e^{6x} - e^{2x} - 1 + e^{-4x})}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{4}(e^{6x} - e^{2x} - 1 + e^{-4x}) &= \frac{1}{4}\left(-1 + \sum_{n\geq 0} \frac{6^n x^n}{n!} - \frac{2^n x^n}{n!} + \frac{(-4)^n x^n}{n!}\right) \\
&= \frac{1}{4}\left(-1 + \sum_{n\geq 0}(6^n - 2^n + (-4)^n)\frac{x^n}{n!}\right),
\end{aligned}$$

we conclude that $\boxed{f(n) = \frac{1}{4}(6^n - 2^n + (-4)^n)}$, for $n \geq 1$.

11 Poset

a. Define a *minimal* element of a poset and a *minimum* element of a poset.

b. Define a *maximal* element of a poset and a *maximum* element of a poset.

c. Define a chain in a poset.

d. Define an antichain in a poset.

e. Define an order filter of a poset.

f. Define a linear extension of a poset.

g. Given two elements $x \leq y$ in a poset, define the interval $[x, y]$.

h. Draw the Hasse diagram of the boolean algebra $B3$ of degree 3 (see Figure 16.2 in Section 16.1).

See also all questions from the last problem set, *except* for lattice (problem 5 and 6).

12 Write your own question (This will be on the exam)

Write a problem related to the techniques and concepts related to set partition, poset or ordinary/exponential generating function. The difficulty of the problem should be comparable to other questions on this sample exam. Write a correct solution key.