Math3250 Combinatorics Week12 Exam Sample

1 Strong induction with sequences

Let $f_0 := 0$, $f_1 := 1$, and

$$f_n := f_{n-1} + f_{n-2} \quad \text{for all } n \geq 2.$$

Let $\gamma := \frac{1}{2}(1+\sqrt{5})$ and $\delta := \frac{1}{2}(1-\sqrt{5})$. Use strong induction[1] to prove that

$$f_n = \frac{1}{\sqrt{5}}(\gamma^n - \delta^n) \quad \text{for all } n \geq 0. \tag{1}$$

2 Integer partitions

Let $k, n \geq 1$.

a. Let $p_k(n)$ denote the number of all partitions of (the integer) n into *exactly* k parts. Prove that $p_k(n)$ is equal to the number of partitions of n in which the largest part is exactly k.

b. Prove that the number of partitions of $n+k$ into exactly k parts is equal to the number of partitions of n into at most k parts is equal. That is, let $p_{\leq k}(n)$ denote the number of partitions of n into at most k parts and prove that $p_k(n+k) = p_{\leq k}(n)$.

3 OGF for integer partitions

For $n \geq 0$, let $p(n)$ denote the number of partitions of the integer n. Recall that

$$\sum_{n=0}^{\infty} p(n)\, x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

a. Fix a positive integer k. Let $p_k(n)$ denote the number of all partitions of (the integer) n into *exactly* k parts. Find the ordinary generating function of the sequence $p_k(n)$.

b. Let $c_0 = 1$ and, if $n \geq 1$, let c_n denote number of partitions of n for which no part appears more than three times. Let $C(x) := \sum_{n=0}^{\infty} c_n x^n$. Find an explicit formula for $C(x)$.

c. Let $d_0 = 1$ and, if $n \geq 1$, let d_n denote number of partitions of n for which no part is divisible by 4. Let $D(x) := \sum_{n=0}^{\infty} d_n x^n$. Find an explicit formula for $D(x)$.

d. Show that the number of partitions of n for which no part appears more than thrice is equal to the number of partitions of n for which no part is divisible by 4.

4 OGF Lucas and other Fibonacci-like numbers

Compute the ordinary generating function for sequences $\{a_n\}$ with the same recurrence relation as the Fibonacci number (with various starting values, for example $a_0 = 5, a_1 = 2$ or $a_1 = 3, a_2 = 4$). Watch Lecture video by Jim Fowler for computing a formula for the Fibonacci numbers.

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[1] You can read or do on your own a proof using ordinary generating function in the book's solution to Exercise 5. I ask you to prove this statement using strong induction because I want you to get better at induction.

5 OGF Tower of Hanoi

In the "Tower of Hanoi" puzzle, you begin with a pyramid of n disks stacked around a center pole, with the disks arranged from the largest diameter on the bottom to the smallest diameter on top. There are also two empty poles that can accept disks. The object of the puzzle is to move the entire stack of disks to one of the other poles, subject to three constraints:

a. Only one disk may be moved at a time.

b. Disks can be placed only on one of the other three poles.

c. A larger disk cannot be placed on a smaller one.

Let a_n be the number of moves required to move the entire stack of n disks to another pole. Here $a_0 = 0$. Compute the ordinary generating function of a_n.

6 OGF from recurrence relation

Consider the sequence defined recursively by $r_0 = 3$, $r_1 = 4$, and $r_n = r_{n-1} + 6r_{n-2}$, for $n \geq 2$. Find a closed form expression for the ordinary generating function $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and use this to find a closed form expression for r_n itself.

7 OGF computation using binomial theorem

a. Write down the definition of $\binom{m}{k}$ for any $m \in \mathbb{R}$ and any $k \in \mathbb{Z}_{>0}$. [2]

b. Write $\sqrt{1-4x}$ as a power series $\sum_{n\geq 0} c_n x^n$.

c. For the exam, practice doing computation using the binomial theorem with a fractional or negative exponent. For more practice, compute the power series form of the expressions in Quick Check Sec 4.3 ($(1-x)^{-m}$ and $\sqrt{1+x}$) and Chapter 4 Exercises 26, 27, 28 ($\sum_{n=1}^{\infty} nx^{n-1}$ and $\frac{1}{\sqrt{1-4x}}$ and $(1-x)(1-x^2)^{1/2}$).

8 OGF and Catalan numbers

Let $b_0 = 1$, and let b_n be the number of triangulations of a regular polygon with $n+2$ vertices for $n \geq 1$. Here are the first few values of b_n.

n	b_n
0	1
1	1
2	2
3	5
4	14
5	42

a. Prove that
$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + b_2 b_{n-3} + \cdots + b_{n-1} b_0.$$

b. Give an explicit formula for the ordinary generating function $B(x) = \sum_{n=0}^{\infty} b_n x^n$ for b_n.

c. Give an explicit formula for b_n.

9 EGF from recurrence relation

Practice with Example 8.17, 8.19 from Section 8.2 of Bona.

[2] Definition 4.14 in Section 4.3.

10 EGF the product formula

Let $f(n)$ be the number of ways to do the following. There are n (distinguishable, off course) children in a classroom. You give an odd number of the children either a red candy or a turquoise candy to eat. I give an odd number of the children either a black marble, a purple marble, or a green marble. The remaining children get nothing. (No child receives more than one item.)

For instance, f(1) = 0 (since there must be at least one child that gets a candy and at least one other child that gets a marble) and f(2) = 12 (two choices for which child gets a candy, two choices for the candy color, and three choices for the marble color).

a. Find a simple formula for the exponential generating function $F(x) = \sum_{n=0}^{\infty} f(n)/n!x^n$ not involving any summation symbols.

b. Find a simple formula for f(n) not involving any summation symbols.

11 Poset

a. Define a *minimal* element of a poset and a *minimum* element of a poset.

b. Define a *maximal* element of a poset and a *maximum* element of a poset.

c. Define a chain in a poset.

d. Define an antichain in a poset.

e. Define an order filter of a poset.

f. Define a linear extension of a poset.

g. Given two elements $x \leq y$ in a poset, define the interval $[x, y]$.

h. Draw the Hasse diagram of the boolean algebra $B3$ of degree 3 (see Figure 16.2 in Section 16.1).

See also all questions from the last problem set, *except* for lattice (problem 5 and 6).

12 Write your own question (This will be on the exam)

Write a problem related to the techniques and concepts related to set partition, poset or ordinary/exponential generating function. The difficulty of the problem should be comparable to other questions on this sample exam. Write a correct solution key.