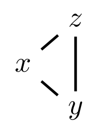
MATH 3250 Combinatorics Week 11 Problem Set

- Turn in handwritten work.
- You are encouraged to work with other people, but write your own solution.

1 Four-element posets

a. Look at the five 3-element posets in Sec 16.1. Explain why 

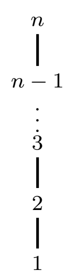
is not the Hasse diagram of a poset.

b. Draw the Hasse diagrams of all 4-element posets. How many are there?

> **Solution:** There are 6 posets with non-connected Hasse diagrams and 10 posets with connected Hasse diagrams.

2 Counting antichains

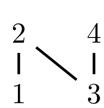
a.) Write down the definition of an *antichain*[1] of a poset.

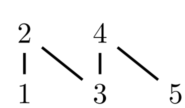
b.) Let $n \geq 1$. Find all antichains of the poset which is the n-element chain, that is, the poset whose Hasse diagram is 

. How many antichains are there?

> **Solution:** The antichains are the empty set and all n singletons, so there are $n + 1$ antichains.

c.) List all antichains of the posets P_4 and P_5 whose Hasse diagrams are shown below. How many antichains are there?[2]



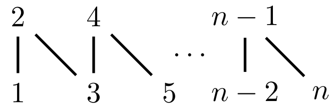


> **Solution:** There are 8 and 13 antichains, including the empty set.

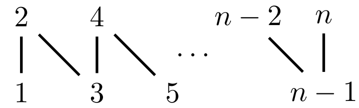
d.) For $n \geq 1$, let a_n be the number of the antichains of the n-element poset P_n whose Hasse diagram is

[1] Sec 16.1, pg 420

[2] You can compare your list with other people's.



if n is odd and



if n is even.

Describe a_n, either recursively or using an explicit formula. **Prove** your answer. [3]

> **Solution:** Let a_n be the number of antichains of the n-element poset P_n. Then $a_1 = 2$ (the empty set and the one-element set) and $a_2 = 3$ (from your answer to part (b)), and
>
> $$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3.$$
>
> To prove this, suppose that $n \geq 3$, and let A be an antichain of P_n. Then either $n \in A$ or $n \notin A$.
>
> If $n \in A$, then $n-1 \notin A$ because $n-1$ and n are comparable. However, none of the numbers $1, \ldots, n-2$ is comparable to n, and there are a_{n-2} ways to pick an antichain of the subposet $[n-2]$.
>
> If $n \notin A$, the number of choices for A is equal to the number of ways to pick an antichain of the subposet $[n-1]$.

3 Antichains and order filters

a. Define a *minimal* element of a poset and a *minimum* element of a poset. [4]

> **Solution:** Let P be a poset. If $x \in P$ such that there no $y \in P$ for which $y < x$, then we say x is a *minimal element* of P.
>
> If $x \leq z$ for all $z \in P$, then we say x is the *minimum element* of P.

b. **Prove** or **disprove** the following statement. If P is a finite poset, then the number of antichains of P is equal to the number of order filters of P. [5]

> **Solution:** The statement is true.
>
> Define a map $\varphi : \{\text{ order filters of } P\ \} \to \{\text{ antichains of } P\ \}$ by
>
> $$\varphi : F \mapsto \phi(F)$$
>
> where $\varphi(\emptyset) = \emptyset$ and $\varphi(F)$ is the set of all minimal elements in F if F is not the empty set.
>
> If $\varphi(F)$ is the empty set or consists of exactly one element, it's clear that $\varphi(F)$ is an antichain. So suppose that $\varphi(F)$ has two or more elements, and let x and y be two distinct elements in $\varphi(F)$. Then x and y are both minimal in F (by definition of φ) so we cannot have $x < y$.
>
> To see that φ is a bijection, let A be an antichain in P. If $A = \emptyset$, then $\emptyset \in \{\text{ order filters of } P\}$ is the unique order filter such that $\varphi(\emptyset) = A$. Otherwise, let
>
> $$F := \{e \in P\colon \text{ there exists } a \in A \text{ where } a \leq e\}.$$
>
> Then F is the unique order filter of P such that $\varphi(F) = A$.

[3] Hint: a_1 and a_2 are computed in part (b); a_3 is computed during class; a_4 and a_5 are computed in part (c). Read the solution to the problem of a child walking up a stairway in Chapter 8's "Solutions to Exercises."

[4] Imitate the language of Section 16.1 (page 419).

[5] Hint: Part (a) of this problem is a hint.

4 Counting filters

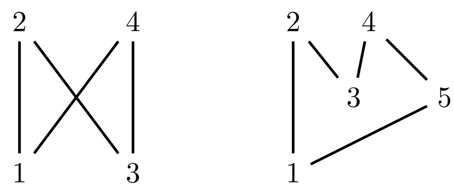
a.) Write down the definition of order ideal[6] and order filter[7] of a poset.

b.) Let $n \geq 1$. Find all order filters of the poset which is the n-element chain, that is, the poset whose Hasse diagram is 

. How many order filters are there?

> **Solution:** The order filters are the empty set and all subsets $[j]$ for $j \in [n]$, so there are $n+1$ order filters.

c.) Let P_4' and P_5' be posets whose Hasse diagrams are shown below.



Let J_4 (resp. J_5) be the set of all order filters of the poset P_4' (resp. P_5'), and define a partial order by inclusion, that is, $F_1 \leq F_2$ iff $F_1 \subset F_2$. Draw the Hasse diagrams of J_4 and J_5 with this partial order. How many order filters are there in J_4 and J_5? [8]

> **Solution:** There are 7 and 10 order filters (respectively), including the empty set.

Note: Parts (d) and (e) were incorrect, so they were removed. We will discuss (i) the correct recurrence relation for these objects and (ii) the correct poset whose order filters/ antichains are satisfy the original recurrence relation, after the exam.

5 Lattice

Moved to the next problem set.

6 Distributive lattice of order filters

Moved to the next problem set.

7 Miscellaneous

i. Write down everyone who contributed to your thought process. Write down Bona's textbook and other written sources you used as well.

[6] Sec 16.2, pg 423

[7] *Order filer* was defined in class during Sec 16.2 lecture, dual to the definition of order ideal.

[8] You can compare your list with other people's. Your answer should be different from Problem 2

ii. Share your work (at least one problem) and thought process with at least one classmate. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. A virtual discussion via Piazza or email is fine if you don't have time to interact in person.

iii. Approximately how much time did you spend on this homework?