Week 11 Mon/Wed Started here

Continue Section 16.2 The Möbius Function of a Poset

Review last week:

The set $U_n := \{n \times n \text{ upper triangular matrices}\}$ forms an algebra with

- addition: usual matrix addition
- multiplication: usual matrix multiplication.

Prop 1 If $F = (f_{i,j})$ & $G = (g_{i,j}) \in U_n$,
$$\begin{array}{cc} \scriptstyle\text{row} & \scriptstyle\text{col} \\ i & j \end{array}$$

then the (i,j) entry of $F.G$ is the sum
$$\begin{array}{cc} \scriptstyle\text{row} & \scriptstyle\text{col} \end{array}$$

because $f_{i,j} = 0 = g_{i,j}$ if $i > j$

$$f_{i,1}\, g_{1,j} + f_{i,2}\, g_{2,j} + \dots + f_{i,n}\, g_{n,j} = f_{i,i}\, g_{i,j} + f_{i,i+1}\, g_{i+1,j} + \dots + f_{i,j}\, g_{j,j}$$
row i col j

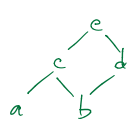
$$= \sum_{k=i}^{j} f_{i,k}\, g_{k,j}$$

$$= \boxed{\sum_{i \le k \le j} f_{i,k}\, g_{k,j}}$$

- The multiplicative identity of U_n is $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ because $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} F = F \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ $\forall$ $F \in U_n$.

Def: Let $Int(P)$ denote the set of all NONEMPTY intervals of P.

Ex

Let $P =$ 

Tasks for students

- List all intervals in the set $Int(P)$
- How many are there?

Note to Ning Wei: This poset is from Stanley EC1 (1st ed) Sec 3.6 pg 261

Answer

[a,a]		[a,c]		[a,e]
	[b,b]	[b,c]	[b,d]	[b,e]
		[c,c]		[c,e]
			[d,d]	[d,e]
				[e,e]

Note: Each of [a,b], [a,d], and [c,d] is equal to the empty interval.

There are twelve (nonempty) intervals.

Def 2 If P is a locally finite poset.

Then the incidence algebra $I(P)$ of P is the set of all functions $f : \text{Int}(P) \longrightarrow \mathbb{R}$.

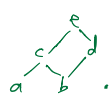
- Addition is defined to be:
  If $f, g \in I(P)$, then $(f+g)$ is the function $\text{Int}(P) \to \mathbb{R}$ defined by:

  $$(f+g)([x,y]) = f([x,y]) + g([x,y]) \quad \forall\ [x,y] \in \text{Int}(P)$$

- Multiplication is defined to be:
  If $f, g \in I(P)$, then $(f \cdot g)$ is the function $\text{Int}(P) \to \mathbb{R}$ defined by: "convolution"

  $$(f \cdot g)([x,y]) = \sum_{x \le z \le y} f([x,z])\, g([z,y]) \qquad (*)$$

Ex

same $P =$  .

① $\mathring{j} : \text{Int}(P) \to \mathbb{R}$
$\mathring{j} : [x,y] \mapsto 0$ for all $[x,y] \in \text{Int}(P)$

② $g : \text{Int}(P) \to \mathbb{R}$
$g : [x,y] \mapsto 5$
for all $[x,y] \in \text{Int}(P)$

③ $h : \text{Int}(P) \to \mathbb{R}$
$h : [a,a] \mapsto 1$
$[a,c] \mapsto 2$
$[a,e] \mapsto \pi$
$[x,y] \mapsto \sqrt{5}$ for all other intervals.

④ **Def** The delta function
$\delta : \text{Int}(P) \to \mathbb{R}$
$$\delta([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x < y. \end{cases}$$

⑤ **Def** The zeta function
$\zeta : \text{Int}(P) \to \mathbb{R}$
$\zeta([x,y]) = 1$ for all $[x,y] \in \text{Int}(P)$

Remark

We can represent $h \in I(P)$ by

1		2		π
	$\sqrt{5}$	$\sqrt{5}$	$\sqrt{5}$	$\sqrt{5}$
		$\sqrt{5}$		$\sqrt{5}$
			$\sqrt{5}$	$\sqrt{5}$
				$\sqrt{5}$

and represent the delta function $\delta \in I(P)$ by

1		0		0
	1	0	0	0
		1		0
			1	0
				1

These look like 5×5 upper triangular matrices!

Prop δ is the multiplicative identity in $I(P)$.

Pf If $f \in I(P)$, then
$$(f \cdot \delta)[x,y] = \sum_{x \le z \le y} f([x,z])\,\delta([z,y])$$
$$= f([x,y])\,\delta([y,y])$$ because $\delta([z,y])$ is nonzero iff $z = y$
$$= f([x,y]) \quad \square$$

$\boxed{\text{Prop 3}}$ If P is finite with n elements, the incidence algebra $I(P)$ is "isomorphic" to the algebra U_n of $n \times n$ upper triangular matrices.

Proof Label the elts of P by $t_1, t_2, \dots, t_n$ so that $\underbrace{t_i \le_P t_j}_{\text{partial order in } P}$ implies $\underbrace{i \le j}_{\text{usual ordering of } [n]}$.

(Note: this is equivalent to fixing a linear extension of P)

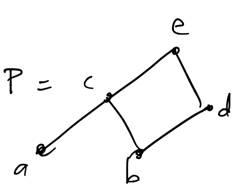
Define a map $\varphi : I(P) \longrightarrow U_n$

by $\varphi : f \longmapsto M$

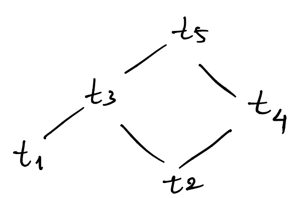
where $M = (m_{ij})_{1 \le i,j \le n}$ s.t

$$m_{i,j} = \begin{cases} f([t_i, t_j]) & \text{if } t_i \le_P t_j \\ 0 & \text{otherwise} \end{cases}$$

Ex

$P =$ 

Choose a linear extension (It's not important which one)



Then $\varphi(f) = \begin{bmatrix} f(1,1) & 0 & f(13) & 0 & f(14) \\ 0 & f(22) & 23 & 24 & 25 \\ 0 & 0 & f(33) & 0 & 35 \\ 0 & 0 & 0 & f(44) & 45 \\ 0 & 0 & 0 & 0 & f(55) \end{bmatrix}$

Note: $\left.\begin{matrix} m_{1,2} \\ m_{1,4} \\ m_{3,4} \end{matrix}\right\} = 0$ because $[t_1, t_2]$, $[t_1, t_4]$, $[t_3, t_4]$ is the empty interval.

• **Note 1** Multiplication in $I(P)$ is "the same" as matrix multiplication in U_n:

Given $f, g \in I(P)$, define matrices $F = \varphi(f) = (m_{ij})$ and $G = \varphi(g) = (h_{ij})$.

(For simplicity, define $f([x,y]) := 0$ and $g([x,y]) := 0$ if $[x,y]$ is empty)

Then the (i,j) entry of $F \cdot G$, by $\boxed{\text{Prop 1}}$, is equal to

$$\sum_{i \le k \le j} m_{ik} h_{kj} = \sum_{i \le k \le j} f([t_i, t_k])\, g([t_k, t_j]) \quad \text{by def of } m_{ij} \text{ \& } h_{ij}.$$

$$= \sum_{t_i \le t_k \le t_j} f([t_i, t_k])\, g([t_k, t_j]) \quad \text{since } f([x,y]) = 0 = g([x,y]) \text{ if } [x,y] \text{ is the empty interval}$$

done on Friday

$$= (f \cdot g)([t_i, t_j]) \quad \text{by eq } (*) \text{ in } \boxed{\text{Def 2}}$$

This shows that $\underbrace{\varphi(f)}_{F} \overset{\text{matrix mult}}{\cdot} \underbrace{\varphi(g)}_{G} = \varphi(f \underset{\text{convolution}}{\cdot} g)$

• **Note 2** Recall from earlier:

- The multiplicative identity in U_n is the identity matrix $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

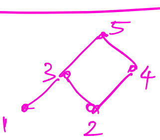
- The multiplicative identity in $I(P)$ is

$$\delta : \mathrm{Int}(P) \to \mathbb{R}$$
$$[x,y] \mapsto \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x < y. \end{cases}$$

Check that $\varphi(\delta) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

$\boxed{\text{End of Prop 3}}$

$\boxed{\text{Task for student}}$

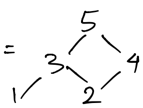
For $P =$  prove that $\varphi(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Week 11 Mon / Wed ended here

— Week 11 Friday started here —

Today: Let P be a locally finite poset.

Def The zeta function $\zeta \in I(P)$ is defined by $\zeta([x,y]) := 1$ for all $[x,y] \in \mathrm{Int}(P)$.

E.g. $P =$ 

Let $Z := \varphi(\zeta)$. Then $Z = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}$

Note:

$$Z^2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

elts in the interval $[1,5]$: 1, 3, 5

elts in the interval $[2,5]$: 2, 3, 4, 5

in $[4,5]$

Prop 1 $\zeta^2([s,u]) =$ # elements between s and u (including s and u).

Pf $\zeta^2([s,u]) = \sum_{s \le t \le u} 1$

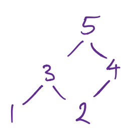
$=$ # elts t s.t. $s \le t \le u$

$=$ # elts in the interval $[s,u]$.

Def A multichain in a poset P is a sequence $(a_1, a_2, \dots, a_m)$ of elts in P satisfying $a_1 \le a_2 \le \dots \le a_m$.

"Note the inequalities are not strict, unlike in the def of chains".

Def The length of a chain/multichain is the # elements minus 1.

$P =$ 

Multichains of length 2

starting at 2 and ending at 5: $2 \le 2 \le 5$, $2 \le 3 \le 5$, $2 \le 4 \le 5$, $2 \le 5 \le 5$

Note: $[2,5] = \{2, 3, 4, 5\}$

Task for students

starting at 1 and ending at 5: $1 \le 1 \le 5$, $1 \le 3 \le 5$, $1 \le 5 \le 5$ — $[1,5] = \{1, 3, 5\}$

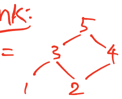
Starting at 4 and ending at 5: $4 \le 4 \le 5$, $4 \le 5 \le 5$ — $[4,5] = \{4, 5\}$

Rem 2 The map $\{\text{multichains of length two } x = x_0 \le x_1 \le x_2 = y\} \to \{\text{elements in interval } [x,y]\}$, $x_1 \mapsto x_1$

is a bijection

Prop 3 By Prop 1 and Rem 2, $\zeta^2([x,y]) = \#\left\{\begin{array}{c}\text{multichains of length two}\\ x = x_0 \le x_1 \le x_2 = y\end{array}\right\}$

Lemma 4 For $k \geqslant 1$, $\displaystyle\sum_{z \in [x,y]} \zeta^{k-1}([x,z])\,\zeta([z,y]) = \zeta^k([x,y])$

Think: P = 

$Z = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & 0 & & 1 & 1 \\ & & & & 1 \end{bmatrix}$

Proof of Lemma 4 Let $Z = \varphi(\zeta)$

The (i,j) entry of Z^2 is

$$\sum_{i \le h \le j} Z_{i,h} Z_{h,j} = \sum_{x \le z \le y} \zeta([x,z])\,\zeta([z,y])$$

By induction on the exponent of Z,
the (i,j) entry of Z^k is

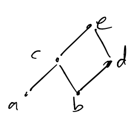
$$\sum_{i \le h \le j} (Z)^{k-1}_{i,h}\, Z_{h,j} = \sum_{x \le z \le y} \zeta^{k-1}([x,z])\,\zeta([z,y])$$

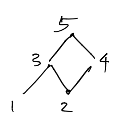
Prop 16.12 Let $x \le y$ be elts of P. Let $k \geqslant 1$.
Then the # of multichains (of length k)
$x = x_0 \le x_1 \le \cdots \le x_k = y$ is equal to $\zeta^k([x,y])$.

——— week 11 Friday ended here ———

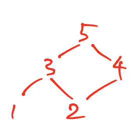
Con't Sec 16.2 Möbius function

Review: • Choose a linear extension of a poset P





• The matrix $Z = (Z_{i,j})$ for the ζ function
has entries $Z_{i,j} = \begin{cases} 1 & \text{if } [i,j] \text{ is a non-empty interval} \\ 0 & \text{o/w} \end{cases}$

P = 

$Z = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & 0 & & 1 & 1 \\ & & & & 1 \end{bmatrix}$ Task for Students

• Prop 1: $(Z^2)_{i,j}$ = # elts in the interval $[i,j]$

• **Lemma 4** $\displaystyle\sum_{x \le z \le y} \zeta^{k-1}([x,y])\,\zeta([z,y]) = \zeta^k([x,y])$.

Prop 16.12 Let $x \le y$ be elts of P. Let $k \geqslant 1$.

Then the # of multichains (of length k)

$x = x_0 \le x_1 \le \cdots \le x_k = y$ is equal to $\zeta^k([x,y])$.

Pf We prove this by induction on k.

$\zeta^1([x,y]) = \zeta([x,y]) \overset{\text{def}}{=} 1$, and the only possible multichain $x = x_0 \le x_1 = y$ is (x,y).

Suppose that the statement is true for all positive integers less than k.

Let $x = x_0 \le x_1 \le \cdots \le x_k = y$ be a multichain of length k.

Then $x_{k-1} = z$ for some $z \in [x,y]$.

By the inductive hypothesis, the number of multichains of length $k-1$ between x and z is

$$\zeta^{k-1}([x,z])$$

and the number of multichains $z \le y$ of length 1 is

$$\zeta([x,y]) = 1.$$

So the # of possibilities for a multichain $x = x_0 \le x_1 \le \cdots \le x_k = y$ of length k is

$$\sum_{z \in [x,y]} \zeta^{k-1}([x,z])\,\zeta([x,y]).$$

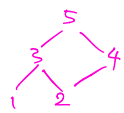
By Lemma 4, this expression is equal to $\zeta^k([x,y])$ ☒

Consider the function $\zeta - \delta \in \underset{\sim}{I(P)}$

incidence algebra, the sets of all functions $\mathrm{Int}(P) \to \mathbb{R}$.

Then $(\zeta-\delta)([x,y]) = \zeta([x,y]) - \delta([x,y]) = \begin{cases} 1-0=1 & \text{if } x<y \\ 1-1=0 & \text{if } x=y \end{cases}$

Example:

$P =$ 

$$\varphi(\zeta-\delta) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & 0 & & 1 & 1 \\ & & & & 1 \end{bmatrix} - \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & 0 & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} =: ZD$$

$$(ZD)^2 = \begin{bmatrix} 0 & & 1 & & 1 \\ & 0 & 1 & 1 & 1 \\ & & 0 & & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \begin{bmatrix} 0 & & 1 & & 1 \\ & 0 & 1 & 1 & 1 \\ & & 0 & & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 2 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$$

— # elts between 1 & 5 (strict)

— # elts between 2 & 5 (strict)

— # elts between 4 & 5 (strict)

Prop I (Compare with Prop 1)

(i) $(\zeta-\delta)^1([x,y]) = \#$ chains $x=x_0<x_1=y$ of length 1

Pf If $x \le y$, then # chains $x=x_0<x_1=y$ of length 1 is $\begin{cases} 1 & \text{if } x<y \\ 0 & \text{if } x=y. \end{cases}$

(ii) $(\zeta-\delta)^2([x,y]) = \#$ chains $x=x_0<x_1<x_2=y$ of length 2

$= \#$ elts in $[x,y]$ not counting x and y.

Lemma IV (Compare with Lemma 4)

$$\sum_{z\in[x,y]} (\zeta-\delta)^{k-1}([x,z]) \cdot (\zeta-\delta)([z,y]) = (\zeta-\delta)^k([x,y])$$

(Exercise)

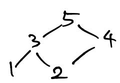
Prop 16.13 (Compare with Prop 16.12) Let $k \geqslant 1$

$$(\zeta-\delta)^k([x,y]) = \sum_{x=x_0<x_1<\cdots<x_k=y} 1,$$ that is

$(\zeta-\delta)^k([x,y])$ is the # of chains of length k which start at x and end in y.

<u>Recap</u>: Functions delta δ multiplicative identity, zeta ζ, and $\zeta - \delta$ in $I(P)$.

Q: Find the inverse of the zeta function.

<u>Ex</u> $P =$ 

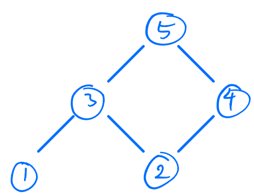
$$Z = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & 1 & 0 & 1 & 1 & & & & 0 \\ & 1 & 1 & 1 & 1 & & 1 & & & \\ & & 1 & 0 & 1 & & & 1 & & \\ & & & 1 & 1 & & & & 1 & \\ 0 & & & & 1 & 0 & & & & 1 \end{array}\right] \rightsquigarrow \begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \\ R_3 - R_4 \\ R_4 - R_5 \\ R_5 \end{array} \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ & & & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ & & & & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow R_2 - R_4 \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

type "rref" $\longrightarrow$

$$Z^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ & 1 & -1 & -1 & 1 \\ & & 1 & 0 & -1 \\ 0 & & & 1 & -1 \\ & & & & 1 \end{bmatrix}$$



Question Does the zeta function ζ of P have an inverse?

If P is finite, Z^{-1} exists. What does Z^{-1} look like?

(Stay tuned)