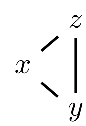
MATH 3250 Combinatorics Week 11 Problem Set

- Turn in handwritten work.
- You are encouraged to work with other people, but write your own solution.

1 Four-element posets

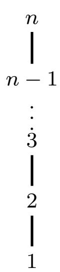
a. Look at the five 3-element posets in Sec 16.1. Explain why 

is not the Hasse diagram of a poset.

b. Draw the Hasse diagrams of all 4-element posets. How many are there?

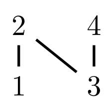
2 Counting antichains

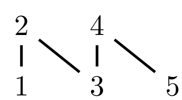
a.) Write down the definition of an *antichain*[1] of a poset.

b.) Let $n \geq 1$. Find all antichains of the poset which is the n-element chain, that is, the poset whose Hasse diagram is 

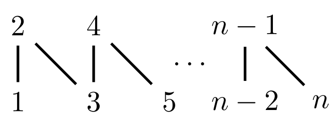
. How many antichains are there?

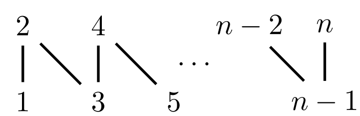
c.) List all antichains of the posets P_4 and P_5 whose Hasse diagrams are shown below. How many antichains are there?[2]





d.) For $n \geq 1$, let a_n be the number of the antichains of the n-element poset P_n whose Hasse diagram is



if n is odd and


if n is even.

Describe a_n, either recursively or using an explicit formula. **Prove** your answer. [3]

3 Antichains and order filters

a. Define a *minimal* element of a poset and a *minimum* element of a poset. [4]

b. **Prove** or **disprove** the following statement. If P is a finite poset, then the number of antichains of P is equal to the number of order filters of P. [5]

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[1] Sec 16.1, pg 420

[2] You can compare your list with other people's.

[3] Hint: a_1 and a_2 are computed in part (b); a_3 is computed during class; a_4 and a_5 are computed in part (c). Read the solution to the problem of a child walking up a stairway in Chapter 8's "Solutions to Exercises."

[4] Imitate the language of Section 16.1 (page 419).

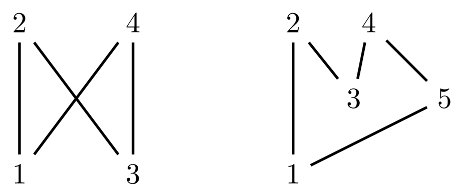
[5] Hint: Part (a) of this problem is a hint.

4 Counting filters

a.) Write down the definition of order ideal[6] and order filter[7] of a poset.

b.) Let $n \geq 1$. Find all order filters of the poset which is the n-element chain, that is, the poset whose Hasse diagram is $\begin{array}{c} n \\ \vdots \\ 2 \\ | \\ 1 \end{array}$. How many order filters are there?

c.) Let P_4' and P_5' be posets whose Hasse diagrams are shown below.



Let J_4 (resp. J_5) be the set of all order filters of the poset P_4' (resp. P_5'), and define a partial order by inclusion, that is, $F_1 \leq F_2$ iff $F_1 \subset F_2$. Draw the Hasse diagrams of J_4 and J_5 with this partial order. How many order filters are there in J_4 and J_5? [8]

Note: Parts (d) and (e) were incorrect, so they were removed. We will discuss (i) the correct recurrence relation for these objects and (ii) the correct poset whose order filters/ antichains are satisfy the original recurrence relation, after the exam.

5 Lattice

Moved to the next problem set.

6 Distributive lattice of order filters

Moved to the next problem set.

7 Miscellaneous

i. Write down everyone who contributed to your thought process. Write down Bona's textbook and other written sources you used as well.

ii. Share your work (at least one problem) and thought process with at least one classmate. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. A virtual discussion via Piazza or email is fine if you don't have time to interact in person.

iii. Approximately how much time did you spend on this homework?

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[6] Sec 16.2, pg 423

[7] *Order filer* was defined in class during Sec 16.2 lecture, dual to the definition of order ideal.

[8] You can compare your list with other people's. Your answer should be different from Problem 2