MT3250 COMBINATORICS WEEK10 PROBLEM SET

YOUR PREFERRED FIRST AND LAST NAME

Credit: Write down everyone who contributed to your thought process. Write down Bona's textbook and other written sources you used as well.

Instruction (please comment out or erase).

- You are encouraged to work with other people, but this week your must write your own solution.
- Because the Calculus 2 computation may be tedious to type, you can submit handwritten work at the beginning of class (or type your work, if you prefer). If you have to take partial fractions or a complicated derivative or antiderivative, you may use a computing tool (or not), but do the rest of your work by hand.
- You should check your computation with a computing tool like WolframAlpha or Mathematics, etc (free for UConn students).

1. Recurrence relations and generating functions

Let $a_1 = 1$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 1$.[1]

(1) Compute a simple explicit formula for a generating function (either ordinary or exponential) of a_n.
(2) Compute a closed-form (no summation) formula for a_n.
Uncomment for a hint:

Answer. An explicit formula for the (pick one: ordinary/exponential) generating function is

$$\boxed{insert\ formula\ in\ terms\ of\ x}.$$

An explicit formula for a_n is $\boxed{insert\ formula\ in\ terms\ of\ n}$. □

[1]Compute a_3, a_4, and a_5 and observe that this is not the Fibonacci sequence.

2. Binary trees

A *binary tree* is a rooted tree where each vertex has a left subtree and a right subtree (which may be empty). In other words, a binary tree is a rooted tree where each vertex has at most 2 children.[2] See slide 18 http://www-math.mit.edu/~rstan/transparencies/china.pdf for the 5 binary trees with $n = 3$ vertices.

Let $b_0 = 1$, and let b_n be the number of binary trees with n vertices for $n \geq 1$. Here are the first few values of b_n.

n	b_n
0	1
1	1
2	2
3	5
4	14
5	42

(1) Prove that

$$b_n = b_0 b_{n-1} + b_1 b_{n-2} + b_2 b_{n-3} + \cdots + b_{n-1} b_0.$$

Uncomment for hints:

(2) Compute the ordinary generating function $B(x) = \sum_{n=1}^{\infty} b_n x^n$ for b_n.

Uncomment for hints:

3. Permutations

Let $d_0 = 1$ and, if $n \geq 1$, let d_n be the number of bijections on $[n]$ such that, for all $i \in [n]$, it sends i to another number. Then d_n satisfies the recurrence (you are not asked to prove this)

$$d_n = n\, d_{n-1} + (-1)^n. \tag{1}$$

a. Use (1) to compute the *exponential* generating function $D(x) = \sum_{n=1}^{\infty} d_n \frac{x^n}{n!}$ for d_n. (Hint: Follow the examples in Sec 8.2.1 Recurrence Relations and Exponential Generating Functions).

b. Use it to prove that

$$d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

Uncomment for a hint:

[2]Note that this definition is different from the *full* binary trees described during lecture.

4. Harmonic numbers

a. Compute by hand the Taylor series for $\ln(\frac{1}{1-x})$ centered at 0. Read your Calculus textbook or watch lecture videos explaining this specific problem.

Uncomment for a hint:

b. Let $h_1 = 0$ and

$$h_{n+1} = (n+1)\ h_n + n! \ \text{ for all } n \geq 0.$$

Compute an explicit formula for a generating function (either ordinary or exponential) of h_n. Use it to prove the formula

$$h_n = n! \sum_{k=1}^{n} \frac{1}{k}.$$

Uncomment for a hint:

Answer. (1) The Taylor series for $\ln\left(\frac{1}{1-x}\right)$ is

$$\boxed{\sum_{n=}^{\infty} \ insert \ x^n}$$

(2) An explicit formula for the (pick one: ordinary/exponential) generating function is

$$\boxed{insert\ formula\ in\ terms\ of\ x}.$$

An explicit formula for a_n is $\boxed{insert\ formula\ in\ terms\ of\ n}$.

□

5. Section 8.2.2: The product formula

Given a classroom with n students ($n \geq 0$), a teacher divides the students into three groups[3] A, B, C so that A has an odd number of people and B has an even number of people (no restriction for C). The teacher then asks each group to form a line.

Let f_n be the number of ways to do this. Find a generating (either ordinary or exponential) function $F(x) = \sum_{n=0}^{\infty} f_n\, x^n$ and use it to find a closed form formula for f_n.

Uncomment for a hint:

[3] groups can be empty

6. Miscellaneous

i. Share your work (at least one problem) and thought process with at least one classmate. Ask them to share their thought process as well. Write down their names and briefly summarize your interactions. A virtual discussion via Piazza or email is fine if you don't have time to interact in person.
ii. Approximately how much time did you spend on this homework?