First & Last Name: ______________________________ Student ID: ________________

Math 2310 Multivariable Calculus III Quiz 4 version b

Instructions: No notes or calculators are allowed. Each page is worth the same amount of points.

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1. Consider the vector field $\mathbf{F} = \langle 7x, 3y \rangle$.

a.) First, demonstrate that $\mathbf{F}$ is conservative (using the Test for Conservative Vector Fields).

b.) Then find its scalar potential function $\varphi(x, y)$. (Show all work on this paper.)

Solution: (Taken from MML Sec 17.3 Problem 3. For more practice, do Problems 2, 5, 6.)

Let $\mathbf{F} = \langle f, g \rangle = \langle 7x, 3y \rangle$
a.) The Test for Conservative Vector Fields in $\mathbb{R}^2$ has a single condition $f_y = g_x$.
We checked that

$$f_y = 0$$
$$g_x = 0$$

So $\mathbf{F}$ is indeed conservative.

b.) We want to find a function $\varphi(x, y)$ such that $\varphi_x = f = 7x$ and $\varphi_y = g = 3y$.

1. Integrate $\varphi_x = f$ with respect to x to obtain φ, which includes an arbitrary function $c(y)$:

$$\varphi = \int 7x \; dx = 7\frac{x^2}{2} + c(y)$$

2. Compute φ_y (using the previous item):

$$\varphi_y = c'(y),$$

then equate this φ_y to g to obtain an expression for $c'(y)$:

$$c'(y) = 3y$$

3. Integrate $c'(y)$ (with respect to y) to obtain $c(y)$:

$$c(y) = 3\frac{y^2}{2} + d$$

and we can take d=0.

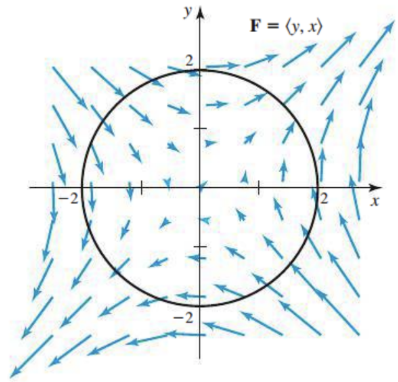
Putting it all together, a potential function is

$$\varphi = \boxed{7\frac{x^2}{2} + 3\frac{y^2}{2}}$$

Optional: do a confidence check by computing the gradient of the function φ you found in part (b).

2. Consider the vector field $\mathbf{F} = \langle y, x \rangle$ shown below. Compute the outward flux across the portion of the circle $x^2 + y^2 = 4$ that is in the third quadrant. (Show all work on this paper.)

Hint: The quarter circle in the third quadrant can be parametrized by $\mathbf{r} = \langle 2\cos t, 2\sin t \rangle$ for $\pi \le t \le \frac{3\pi}{2}$.



Solution: (Taken from MML Sec 17.2 Problem 11. For more practice, do also Problem 12.)
Let

$$\mathbf{F} = \langle f, g \rangle = \langle y, x \rangle,$$

and

$$C : \mathbf{r} = \langle x(t), y(t) \rangle = \langle 2\cos t, 2\sin t \rangle \text{ for } \pi \le t \le \frac{3\pi}{2}.$$

So

$$f(t) = 2\sin t, g(t) = 2\cos t,$$
$$x'(t) = -2\sin t, y'(t) = 2\cos t$$

Then the flux of $\mathbf{F}$ across C is

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_a^b (f(t)\; y'(t) - g(t)\; x'(t))\; dt \\
&= \int_\pi^{\frac{3\pi}{2}} 2\sin t\; 2\cos t - 2\cos t\; (-2\sin t)\; dt \\
&= \int_\pi^{\frac{3\pi}{2}} 8\;\; \cos t\;\; \sin t\; dt \\
&= \int_\pi^{\frac{3\pi}{2}} 4\sin(2t)\; dt \\
&\left(\text{use the identity} \quad \sin\theta\cos\theta = \frac{\sin(2\theta)}{2}, \text{ or use u-substitution with } u = \cos(t) \text{ or } \sin(t) \right) \\
&= 4\left[-\frac{\cos(2t)}{2} \right|_\pi^{\frac{3\pi}{2}} \\
&= -\frac{4}{2}(\cos(3\pi) - \cos(2\pi)) \\
&= -\frac{4}{2}(-1 - 1) = 4
\end{aligned}$$