

Math 2310 Multivariable Calculus III Quiz 4 version a**Instructions:** No notes or calculators are allowed. Each page is worth the same amount of points.**1.** Consider the vector field $\mathbf{F} = \langle 5x, 9y \rangle$.a.) First, demonstrate that \mathbf{F} is conservative (using the Test for Conservative Vector Fields).b.) Then find its scalar potential function $\varphi(x, y)$. (Show all work on this paper.)**Solution:** (Taken from MML Sec 17.3 Problem 3. For more practice, do Problems 2, 5, 6.)Let $\mathbf{F} = \langle f, g \rangle = \langle 5x, 9y \rangle$ a.) The Test for Conservative Vector Fields in \mathbb{R}^2 has a single condition $f_y = g_x$.

We checked that

$$f_y = 0$$

$$g_x = 0$$

So \mathbf{F} is indeed conservative.b.) We want to find a function $\varphi(x, y)$ such that $\varphi_x = f = 5x$ and $\varphi_y = g = 9y$.1. Integrate $\varphi_x = f$ with respect to x to obtain φ , which includes an arbitrary function $c(y)$:

$$\varphi = \int 5x \, dx = 5\frac{x^2}{2} + c(y)$$

2. Compute φ_y (using the previous item):

$$\varphi_y = c'(y),$$

then equate this φ_y to g to obtain an expression for $c'(y)$:

$$c'(y) = 9y$$

3. Integrate $c'(y)$ (with respect to y) to obtain $c(y)$:

$$c(y) = 9\frac{y^2}{2} + d$$

and we can take $d=0$.

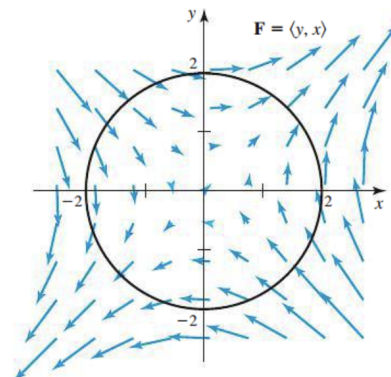
Putting it all together, a potential function is

$$\varphi = \boxed{5\frac{x^2}{2} + 9\frac{y^2}{2}}$$

Optional: do a confidence check by computing the gradient of the function φ you found in part (b).

2. Consider the vector field $\mathbf{F} = \langle y, x \rangle$ shown below. Compute the outward flux across the portion of the circle $x^2 + y^2 = 4$ that is in the third quadrant. (Show all work on this paper.)

Hint: The quarter circle in the third quadrant can be parametrized by $\mathbf{r} = \langle 2 \cos t, 2 \sin t \rangle$ for $\pi \leq t \leq \frac{3\pi}{2}$.



Solution: (Taken from MML Sec 17.2 Problem 11. For more practice, do also Problem 12.)

Let

$$\mathbf{F} = \langle f, g \rangle = \langle y, x \rangle,$$

and

$$C : \mathbf{r} = \langle x(t), y(t) \rangle = \langle 2 \cos t, 2 \sin t \rangle \text{ for } \pi \leq t \leq \frac{3\pi}{2}.$$

So

$$\begin{aligned} f(t) &= 2 \sin t, g(t) = 2 \cos t, \\ x'(t) &= -2 \sin t, y'(t) = 2 \cos t \end{aligned}$$

Then the flux of \mathbf{F} across C is

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{n} \, ds &= \int_a^b (f(t) y'(t) - g(t) x'(t)) \, dt \\ &= \int_{\pi}^{\frac{3\pi}{2}} 2 \sin t \, 2 \cos t - 2 \cos t \, (-2 \sin t) \, dt \\ &= \int_{\pi}^{\frac{3\pi}{2}} 8 \cos t \sin t \, dt \\ &= \int_{\pi}^{\frac{3\pi}{2}} 4 \sin(2t) \, dt \\ &\left(\text{use the identity } \sin \theta \cos \theta = \frac{\sin(2\theta)}{2}, \text{ or use u-substitution with } u = \cos(t) \text{ or } \sin(t) \right) \\ &= 4 \left[-\frac{\cos(2t)}{2} \right]_{\pi}^{\frac{3\pi}{2}} \\ &= -\frac{4}{2} (\cos(3\pi) - \cos(2\pi)) \\ &= -\frac{4}{2} (-1 - 1) = 4 \end{aligned}$$