2. If the density of the surface which is a portion of the paraboloid $z = x^2 + y^2$ $(0 \le z \le 4)$ is given by $f(x, y, z) = 32\,(x^2 + y^2)$, determine the total mass on the surface.

Note: Express the answer as a simplified integral in polar coordinates. You don't need to evaluate the integral.

Solution: Reference: Section 17.6, Theorem 17.14: Evaluation of surface integrals of scalar-valued functions on explicitly defined surfaces.

See solutions of Problem 8 on Spring 2021 Final Exam.

Ref: See Sec 17.6 Example 6, MML # 6

Mass on a surface S is the surface integral $\iint_S \begin{pmatrix}\text{density}\\ \text{function}\end{pmatrix} dS = \iint_S f(x,y,z)\, dS$

Since the surface is explicitly defined by $z = \underbrace{x^2 + y^2}_{\text{let } g(x,z) = x^2 + y^2}$, $0 \le z \le 4$,

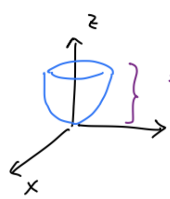
we will use the double integral

THEOREM 17.14 Evaluation of Surface Integrals of Scalar-Valued Functions on Explicitly Defined Surfaces

Let f be a continuous function on a smooth surface S given by $z = g(x, y)$, for (x, y) in a region R. The surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}\, dA.$$

If $f(x, y, z) = 1$, the surface integral equals the area of the surface.

Sketch of S: 

z is from 0 to 4

Set $z = 4$:
$x^2 + y^2 = 4$

The projection of S on the xy-plane is $R = \{(x,y) : x^2 + y^2 \le 4\} = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$z_x = 2x$, $z_y = 2y$, so $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$

$$\iint_R f(x,y,z)\, dS = \iint_R 32\,\underbrace{(x^2 + y^2)}_{r^2}\sqrt{\underbrace{4x^2 + 4y^2}_{4r^2} + 1}\; dA$$

write the iterated integral in polar because R is a disk (See Sec 16.3 "Double integrals in polar coordinates")

$$= 32\int_0^{2\pi}\int_0^2 r^2\sqrt{4r^2 + 1}\ \overset{\text{extra}}{r}\, dr\, d\theta$$