First & Last Name: ______________________ Student ID: ______________

Math 2310 Multivariable Calculus III Group QUIZ FIVE

Instructions: No notes or calculators are allowed. Please box your final answer.

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1. What is your group number? What are the names of the other people in your group?

2. Find an equation of the plane containing the points $A(2,0,0)$, $B(0,4,0)$, and $D(0,0,8)$. Then write it in the form $z = g(x,y)$ where g is a function of x and y.

Solution: A possible normal vector for the plane is $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 0 \\ -2 & 0 & 8 \end{vmatrix} = \langle 32, 16, 8 \rangle$.

To write an equation of a plane, it is sufficient to know a normal vector and a point on the plane (we can use the point $A(2,0,0)$):

$$\begin{aligned} (\text{normal vector}) \ \cdot \overrightarrow{A(x,y,z)} &= 0 \\ \langle 32, 16, 8 \rangle \ \cdot \langle x-2, y-0, z-0 \rangle &= 0 \\ 32x - 64 + 16y + 8z &= 0 \\ 4x - 8 + 2y + z &= 0 \\ \boxed{z = 8 - 4x - 2y} \end{aligned}$$

$\boxed{\textbf{2.}}$ Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = z\mathbf{i} - z\mathbf{j} + (x^2 - y^2)\mathbf{k}$ and C is the closed triangular path from the point $(2, 0, 0)$, to the point $(0, 4, 0)$, to the point $(0, 0, 8)$, and back to the first point $(2, 0, 0)$.

(Use Stokes' Theorem! You can use your computation from the previous page.)

Solution: Reference: Section 17.7 Example 2

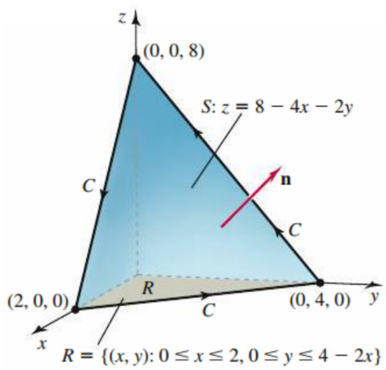

Figure 17.62

➤ Recall that for an explicitly defined surface S given by $z = s(x, y)$ over a region R with $\mathbf{F} = \langle f, g, h \rangle$,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R (-f z_x - g z_y + h)\, dA.$$

In Example 2, $\mathbf{F}$ is replaced with $\nabla \times \mathbf{F}$.

EXAMPLE 2 **Using Stokes' Theorem to evaluate a line integral** Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = z\,\mathbf{i} - z\,\mathbf{j} + (x^2 - y^2)\,\mathbf{k}$ and C consists of the three line segments that bound the plane $z = 8 - 4x - 2y$ in the first octant, oriented as shown in Figure 17.62.

SOLUTION Evaluating the line integral directly involves parameterizing the three line segments. Instead, we use Stokes' Theorem to convert the line integral to a surface integral, where S is that portion of the plane $z = 8 - 4x - 2y$ that lies in the first octant. The curl of the vector field is

$$\nabla \times \mathbf{F} = \nabla \times \langle z, -z, x^2 - y^2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix} = \langle 1 - 2y, 1 - 2x, 0 \rangle.$$

The appropriate vector normal to the plane $z = 8 - 4x - 2y$ is $\langle -z_x, -z_y, 1 \rangle = \langle 4, 2, 1 \rangle$, which points upward, consistent with the orientation of C. The triangular region R in the xy-plane beneath S is found by setting $z = 0$ in the equation of the plane; we find that $R = \{(x, y): 0 \le x \le 2, 0 \le y \le 4 - 2x\}$. The surface integral in Stokes' Theorem may now be evaluated:

$$\iint_S \underbrace{(\nabla \times \mathbf{F})}_{\langle 1 - 2y,\, 1 - 2x,\, 0 \rangle} \cdot \mathbf{n}\, dS = \iint_R \langle 1 - 2y, 1 - 2x, 0 \rangle \cdot \langle 4, 2, 1 \rangle\, dA \qquad \text{Substitute and convert to a double integral over } R.$$

$$= \int_0^2 \int_0^{4-2x} (6 - 4x - 8y)\, dy\, dx \qquad \text{Simplify.}$$

$$= -\frac{88}{3}. \qquad \text{Evaluate integrals.}$$

The circulation around the boundary of R is negative, indicating a net circulation in the clockwise direction on C (looking from above).

<u>Flux of vector Fields</u> through explicitly defined surfaces S given by $z = g(x, y)$ for $(x, y) \in R$:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F} \cdot < -g_x, -g_y, 1 > dA \quad \text{where } \vec{F}(x, y, z) \text{ is a vector field.}$$

<u>Stokes' Theorem</u>: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$ where the direction of travel for C, the orientation of S, and the direction of $\vec{n}$ are consistent.