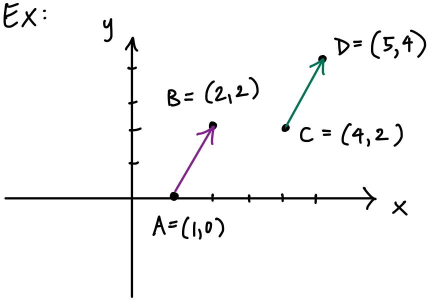
13.1 2D Vectors in the plane

Def (vector)

A vector is a quantity with length (magnitude) and direction.

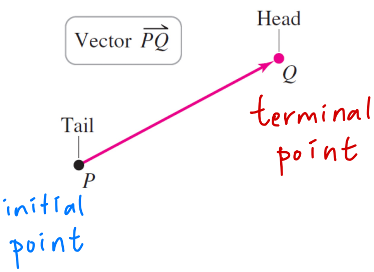
Ex:

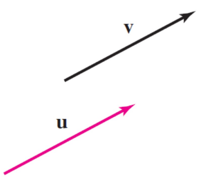


- The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal

bold text

- The zero vector, denoted $\mathbf{0}$ or $\vec{0}$. has length zero and no direction.





Vectors **u** and **v** are *equal* if they have the same length and direction.

Def (scalar multiplication)

Def A **scalar** is a number

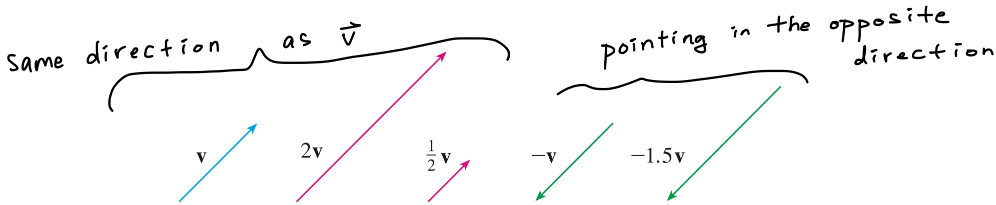
Def If $\vec{v}$ is a vector and c is a number,

the **scalar multiple** $c\vec{v}$ is a vector with ...

- length $|c|$ times length of $\vec{v}$
- (if $c > 0$) same direction as $\vec{v}$

  (if $c < 0$) opposite direction to $\vec{v}$
- If $c = 0$ or $\vec{v} = \vec{0}$, then $c\vec{v} = \vec{0}$.

Scalar multiples of $\vec{v}$



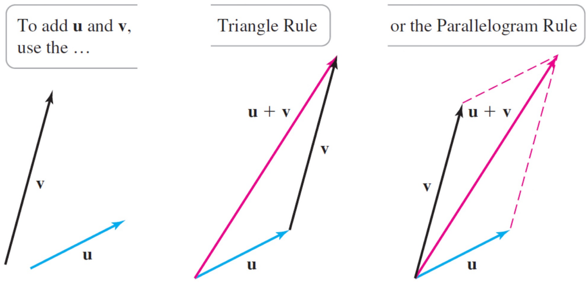
The numbers $1, 2, \frac{1}{2}, -1, -1.5$ "scale" the vector $\vec{v}$.

That's why they are called scalars.

Note: Two vectors are parallel (having the same or opposite direction) if they are scalar multiples of one another.

Def (vector addition)

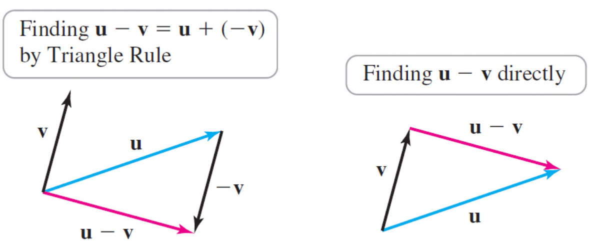
To get the vector $\vec{u}+\vec{v}$,
put the initial point of v at the terminal point of u.



Then $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$

Fact: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

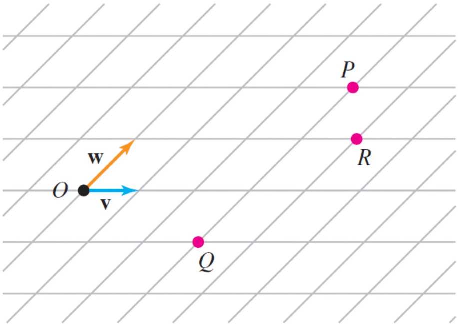
Vector subtraction:



Ex: Write the vectors

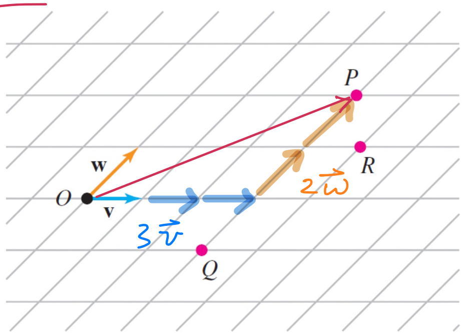
a) $\overrightarrow{OP}$, b) $\overrightarrow{OQ}$, c) $\overrightarrow{QR}$

as sums of scalar multiples (linear combinations) of $\vec{v}$ and $\vec{w}$.

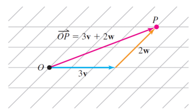


Sol:

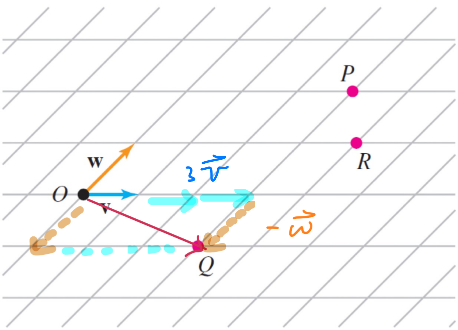
a)



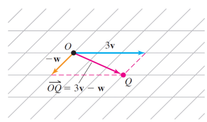
$\overrightarrow{OP} = 3\vec{v} + 2\vec{w}$



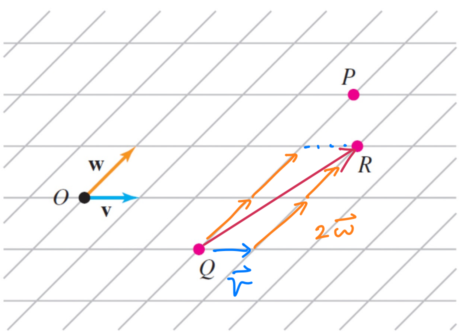
b)



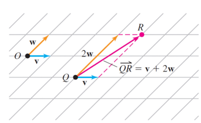
$\overrightarrow{OQ} = 3\vec{v} - \vec{w}$



c)



$\overrightarrow{QR} = \vec{v} + 2\vec{w}$

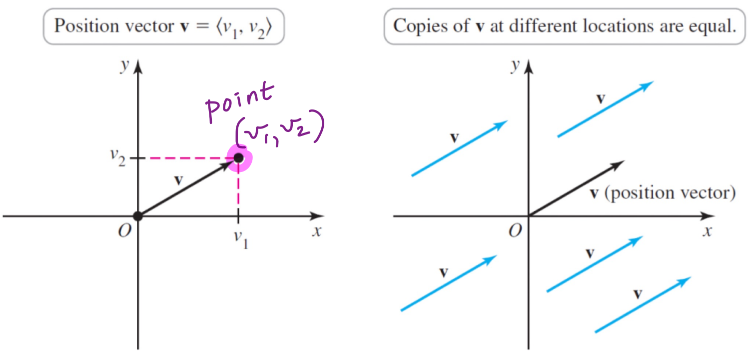


Vector Components

A vector $\vec{v}$ with its tail at the origin and head at the point (a, b) is called a <u>position</u> <u>vector</u> and is written $\langle a, b \rangle$.

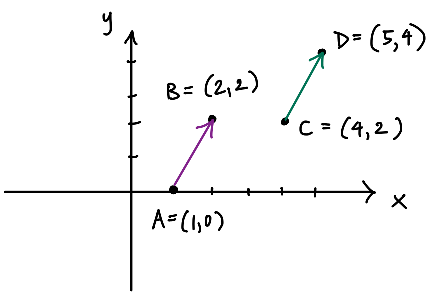
x-component of $\vec{v}$

y-component of $\vec{v}$



Ex:

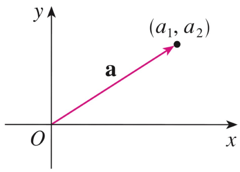
Find a vector with representation given by $\overrightarrow{CD}$



<u>Answer</u> $\langle 5-4, 4-2 \rangle$

$= \langle 1, 2 \rangle$

Magnitude or length of a vector



$\mathbf{a} = \langle a_1, a_2 \rangle$

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is
(or magnitude)

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

Ex: Given the points $P(-3, 4)$ and $Q(6, 5)$, find the components and magnitude of the vector $\overrightarrow{PQ}$.

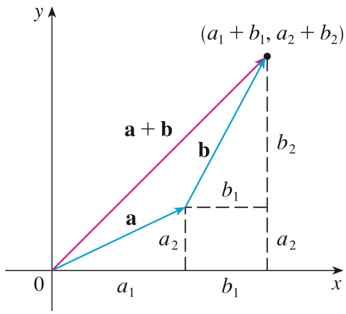
Sol: a) $\overrightarrow{PQ} = \langle 6 - (-3), 5 - 4 \rangle = \langle 9, 1 \rangle$

x-component is 9

y-component is 1

b) Magnitude $|\overrightarrow{PQ}|$ is $\sqrt{9^2 + 1^2} = \sqrt{82}$

Adding vectors using algebra



If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

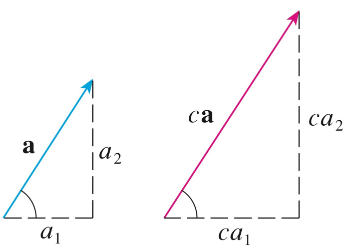
$$\mathbf{a} + \mathbf{b} = \langle \underbrace{a_1 + b_1}_{}, \underbrace{a_2 + b_2}_{} \rangle$$

Add each component of $\vec{a}$ and $\vec{b}$

Ex: If $\mathbf{a} = \langle 4, 0 \quad \rangle$ and $\mathbf{b} = \langle -2, 1 \quad \rangle$,

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \langle 4, 0 \quad \rangle + \langle -2, 1 \quad \rangle \\ &= \langle 4 + (-2), 0 + 1 \quad \rangle = \langle 2, 1 \quad \rangle \end{aligned}$$

Scaling vector using algebra



$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Multiply each component of $\vec{a}$ by c

Ex: If $\mathbf{a} = \langle 4, 0 \quad \rangle$ and $\mathbf{b} = \langle -2, 1 \quad \rangle$,

$$\begin{aligned} 2\mathbf{a} + 5\mathbf{b} &= 2\langle 4, 0 \quad \rangle + 5\langle -2, 1 \quad \rangle \\ &= \langle 8, 0 \quad \rangle + \langle -10, 5 \quad \rangle = \langle -2, 5 \quad \rangle \end{aligned}$$

MML Problem 16

Let $\overline{u} = \langle 1, 1\rangle$, $\overline{v} = \langle 5, -1\rangle$, and $\overline{w} = \langle -4, 0\rangle$. Find the vector $\overline{x}$ that satisfies

$$10\overline{u} - \overline{v} + \overline{x} = 8\overline{x} + \overline{w}.$$

Answer:

$$10\vec{u} - \vec{v} - \vec{w} = 7\vec{x}$$

$$\frac{1}{7}\left(10\vec{u} - \vec{v} - \vec{w}\right) = \vec{x}$$

$$\vec{x} = \frac{1}{7}\left(\langle 10.1, 10.1\rangle - \langle 5, -1\rangle - \langle -4, 0\rangle\right)$$

$$= \frac{1}{7}\left(\langle 10-5+4, 10+1\rangle\right)$$

$$= \frac{1}{7}\langle 9, 11\rangle$$

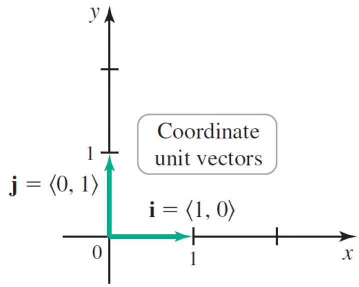
$$= \boxed{\left\langle \frac{9}{7}, \frac{11}{7}\right\rangle}$$

Unit Vectors

Any vector with length 1 is a unit vector.

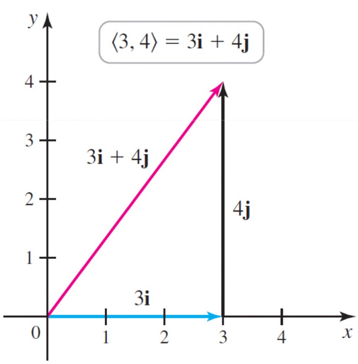
Ex: Coordinate unit vectors $\hat{\imath} = \langle 1,0 \rangle$ and $\hat{\jmath} = \langle 0,1 \rangle$

(also called standard basis vectors)

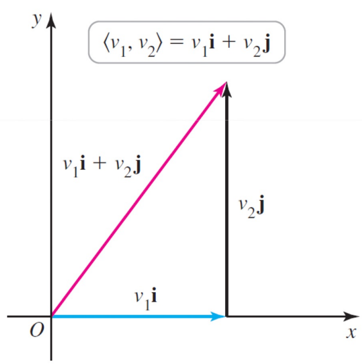


Any 2D vector can be written as a linear combination $a\hat{\imath} + b\hat{\jmath}$ of $\hat{\imath}$ and $\hat{\jmath}$:

Ex:



In general:



Ex: Consider the points $P(1,-2)$ and $Q(6,10)$.

a.) Find two unit vectors parallel to $\overrightarrow{PQ}$.

Sol: $\overrightarrow{PQ} = \langle 6-1,\ 10-(-2)\rangle = \langle 5, 12\rangle$ or $5\hat{\imath} + 12\hat{\jmath}$

Length of $\overrightarrow{PQ}$ is $|\overrightarrow{PQ}| = \sqrt{5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13$

- The unit vector pointing in the same direction as $\overrightarrow{PQ}$ is $\dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{1}{13}\langle 5, 12\rangle = \boxed{\left\langle \dfrac{5}{13}, \dfrac{12}{13}\right\rangle}$

  or $\dfrac{5}{13}\hat{\imath} + \dfrac{12}{13}\hat{\jmath}$

- The unit vector parallel to $\overrightarrow{PQ}$ with the opposite direction is $-\dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \boxed{\left\langle -\dfrac{5}{13}, -\dfrac{12}{13}\right\rangle}$

b.) Find two vectors of length 2 parallel to $\overrightarrow{PQ}$

Sol: Multiply the two unit vectors above by 2:

$$2\left(\frac{5}{13}\hat{\imath} + \frac{12}{13}\hat{\jmath}\right) = \frac{10}{13}\hat{\imath} + \frac{24}{13}\hat{\jmath}$$

$$2\left(-\frac{5}{13}\hat{\imath} - \frac{12}{13}\hat{\jmath}\right) = -\frac{10}{13}\hat{\imath} - \frac{24}{13}\hat{\jmath}$$