Exam 2 Review Questions - Calculus III - Spring 2024

Last edited: March 27, 2024

Exam 2 is on Friday, March 29 in class and will cover sections 15.2-15.7 and 16.1-16.5.

"Shapes Review" Questions:

1. Graphs in $\mathbb{R}^2$: You should know how to sketch the following graphs. Be able to use algebra to find intercepts and intersection points.
   (a) What are equations of lines? Sketch them.
   (b) Equation of a parabola whose y-intercept is c? Sketch it.
   (c) Equation of a circle with radius r centered at the origin? Equation of the upper half of the circle? Equation of the lower half of the circle? Sketch them.
   (d) Sketch the square root function: $y = \sqrt{x}$
2. Graphs in $\mathbb{R}^3$: Be able to visualize the following graphs in order to set up double and triple integrals.
   (a) What are cylinders? A special cylinder is described by the equation $x^2 + y^2 = r^2$. Sketch it.
   (b) What is a general equation of a plane?
   (c) Write an equation of the sphere of radius r centered at the origin, the upper hemisphere, and the lower hemisphere. Sketch them.
   (d) Describe the (closed) ball of radius r centered at the origin. Sketch it.
   (e) Write down the equation of a paraboloid which opens up, whose z-intercept is $(0, 0, a)$. Write down the same paraboloid which opens down.

Answers to the "Shapes Review":

1. Graphs in $\mathbb{R}^2$:
   (a) Lines $y = mx + b$ or $x = c$
   (b) Basic parabolas: $y = ax^2 + c$
   (c) Circle with radius r centered at the origin: $x^2 + y^2 = r^2$
   Solving for y, we get $y = \sqrt{r^2 - x^2}$ (upper semicircle) and $y = -\sqrt{r^2 - x^2}$ (lower semicircle).
   (d) Square root function: $y = \sqrt{x}$
2. Graphs in $\mathbb{R}^3$:
   (a) Cylinders: Graphs whose equations don't involve all of the variables x, y, and z. In particular, you should know about the special cylinder $x^2 + y^2 = r^2$.
   (b) Planes: $ax + by + cz = d$
   Be able to sketch the plane in the first octant when $a, b, c, d > 0$.
   (c) Sphere of radius r centered at the origin: $x^2 + y^2 + z^2 = r^2$
   Solving for z, we get $z = \sqrt{r^2 - x^2 - y^2}$ (hemisphere above the xy-plane) and $z = -\sqrt{r^2 - x^2 - y^2}$ (hemisphere below the xy-plane).
   (d) Ball of radius r centered at the origin: $x^2 + y^2 + z^2 \leq r^2$
   This is the set of all points on the sphere from part (c) plus all of the points inside the sphere.
   (e) Basic circular paraboloid: $z = x^2 + y^2 + a$ (opens up) and $z = a - x^2 - y^2$ (opens down).

Textbook and MML review problems:

To review MML homework, go to your "Gradebook" on Pearson, select "entire course to date" on the drop-down menu, then click on "Review".

1. Textbook Section 15.2 Examples 1 and 2, and MML Section 15.2 Problem 3
2. MML Section 15.3 Problems 1, 4, 7, 8 ; MML Section 15.4 Problems 2, 7, 8
3. MML Section 15.5 Problems 3, 5, 7, 8, 9 ; MML Section 15.6 Problems 1, 3, 5, 6
4. MML Section 15.7 Problems 2, 4, 6, 7
5. MML Section 16.1 Problems 7, 8 ; MML Section 16.2 Problems 5, 6, 8, 11, 12, 13.
6. MML Section 16.3 Problems 2, 3, 6, 7, 9 ; MML Section 16.4 Problems 3, 5
7. MML Section 16.5 Problems 2, 4, 6, 7

Additional review problems:

1. Let $f(x, y) = 9y - x^2y - y^2 + 6$.
   (a) Compute ∇f.
   (b) Compute the following directional derivatives. (Recall that $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.)
      i. $D_{\mathbf{i}} f(3, -1)$
      ii. $D_{\mathbf{j}} f(3, -1)$
      iii. $D_{\mathbf{w}} f(3, -1)$, where $\mathbf{w}$ is a unit vector in the direction of $\langle -2, 5\rangle$.
   (c) Find the equation of the tangent plane to $f(x, y)$ at the point $(x, y) = (3, -1)$.
   (d) Find a unit vector $\mathbf{u}$ which gives the direction of steepest ascent (fastest increase) for the function $f(x, y)$ at the point $(x, y) = (3, -1)$. Then compute the rate of change of f in this direction.
   (e) Find a unit vector $\mathbf{v}$ which gives the direction of steepest descent (fastest decrease) for the function $f(x, y)$ at the point $(x, y) = (3, -1)$. Then compute $D_{\mathbf{v}} f(3, -1)$.
   (f) (Question Removed)
   (g) Find a nonzero vector in $\mathbb{R}^2$ which gives a direction of no change in the function $f(x, y)$ from the point $(x, y) = (3, -1)$.
   (h) Find the critical points of $f(x, y)$, then classify them using the second derivative test.
   (i) **For this part only**, assume that x, y are both functions of t, u. Set up the chain rule for computing $\dfrac{\partial f}{\partial t}$. Then compute $\dfrac{\partial f}{\partial t}$ if $x = t^3u$ and $y = tu^2$.
2. Find a normal vector to the surface $xy - z^2 = y^2z - 1$ at the point $(7, 3, 2)$. Then find the equation of the tangent plane at this point.
3. Let R be the region in the xy-plane bound by the lines $x + y = 2$, $y - x = 2$, and $y = 5$. Set up a single iterated double integral to compute $\displaystyle\iint_R f(x, y)\, dA$ using the order of integration $dx\, dy$ or $dy\, dx$ (whichever one is appropriate). Then compute the double integral using the function $f(x, y) = x^2$.
4. Reverse the order of integration of $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 3e^{(x^3)}\, dx\, dy$, then evaluate the integral.

5. Consider the region R in the second quadrant, bound the line $y = -x$, the x-axis, and curve $y = \sqrt{25 - x^2}$. (This is a "wedge" contained in the second quadrant.) Write R in set-builder notation using polar coordinates:

$$R = \{(r, \theta) \;:\; \_\_\_\_ \le r \le \_\_\_\_ \text{ and } \_\_\_\_ \le \theta \le \_\_\_\_\}$$

6. Convert the following double integral from Cartesian to polar coordinates. Then evaluate the integral:

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \sin(\pi x^2 + \pi y^2)\, dy\, dx$$

7. Consider the region R in the xy-plane bound by the curves $y = x^2$ and $y = 6x - 8$. Let D be the solid lying above the region R and below the plane $z = x$.
   (a) Use the appropriate double integral to compute the volume of the solid D.
   (b) Use the appropriate triple integral to compute the volume of the solid D.
   (c) Find the average value of the function $f(x, y) = x$ over the region R.

8. Consider the solid below the paraboloid $z = 16 - x^2 - y^2$ and above the xy-plane. A cylindrical hole is cut through this solid using the cylinder $x^2 + y^2 = 4$, resulting in a new solid D. Set up a double integral in polar coordinates for computing the volume of the solid D, then compute the volume.

9. Consider the solid D bound by the sphere $x^2 + y^2 + z^2 = 20$ and the paraboloid $z = x^2 + y^2$ in the first octant. Set up a triple integral in cylindrical coordinates to compute the volume of this solid.

10. Let D be the top half of a ball of radius 3 centered at the origin. Find the average distance of points in D from the origin using the appropriate triple integral in spherical coordinates.

    Hint: Set up a function $f(\rho, \varphi, \theta)$ which gives the distance of the point (ρ, φ, θ) to the origin. Then use the triple integral formula for the average of a function.

Answers to additional review problems:

1. (a) $\langle -2xy,\, 9 - x^2 - 2y \rangle$
   (b) i. 6
   ii. 2
   iii. $-\frac{2}{\sqrt{29}}$
   (c) $z = -11 + 6x + 2y$ or $6x + 2y - z = 11$
   (d) $\mathbf{u} = \langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$,
   max rate of change is $2\sqrt{10}$
   (e) $\mathbf{v} = \langle -\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \rangle$,
   $D_{\mathbf{v}} f(3, -1) = -2\sqrt{10}$
   (f) (Question removed)
   (g) $\langle 1, -3 \rangle$ or any nonzero scalar multiple of this vector
   (h) Critical points: $(0, \frac{9}{2})$, $(3, 0)$, $(-3, 0)$
   $(0, \frac{9}{2})$ is a local maximum, and the other two points are saddle points.
   (i) Chain rule: $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$

   2nd answer: $-7t^6u^4 - 2tu^4 + 9u^2$

2. Normal vector: $\langle -3, 5, 13 \rangle$ (or a nonzero scalar multiple)
   Equation: $-3x + 5y + 13z = 20$

3. Iterated integral: $\displaystyle\int_2^5 \int_{-y+2}^{y-2} f(x, y)\, dx\, dy$
   2nd answer: $\displaystyle\iint_R x^2\, dA = \frac{27}{2}$

4. Iterated integral in reverse order: $\displaystyle\int_0^2 \int_0^{x^2} e^{(x^3)}\, dy\, dx$
   2nd answer: $e^8 - 1$

5. $R = \{(r, \theta) : 0 \le r \le 5 \text{ and } \frac{3\pi}{4} \le \theta \le \pi\}$

6. 1

7. volume 4, average value 3

8. 72π

9. $\displaystyle\int_0^{\pi/2} \int_0^2 \int_{r^2}^{\sqrt{20-r^2}} r\, dz\, dr\, d\theta$

10. $\dfrac{9}{4}$