Math 3230 Abstract Algebra I
Sec 5.3: Examples of group actions

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Abstract Algebra I

Groups acting on elements, subgroups, and cosets

It is often of interest to analyze the action of a group G on its elements, subgroups, or cosets of some fixed $H \leq G$.

Sometimes, the orbits and stabilizers of these actions are algebraic objects you have seen before.

Sometimes a deep theorem has a slick proof via a clever group action. For example, we will see how Cayley's theorem (every group G is isomorphic to a group of permutations) follows immediately once we look at the correct action.

Here are common examples of group actions:

a. G acts on itself by right-multiplication (or left-multiplication).
b. G acts on itself by conjugation.
c. G acts on its subgroups by conjugation (not covered).
d. G acts on the right-cosets of a fixed subgroup $H \leq G$ by right-multiplication.

For each of these, we'll analyze the orbits, stabilizers, and fixed points.

Recall Example 3: Groups acting on themselves by right-multiplication

We've seen how groups act on themselves by right-multiplication (Example 3 in Slides 5.1). While this action is boring (any Cayley diagram is an action diagram!), it leads to a slick proof of Cayley's theorem.

Theorem 2 (Cayley's theorem)

If $|G| = n$, then there is an injective homomorphism $G \hookrightarrow S_n$.

Proof.

The group G acts on itself (that is, $S = G$) by **right-multiplication**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S) \cong S_n\,, \qquad \phi(g) \text{ is the permutation that sends each } x \mapsto xg.$$

There is only one orbit: $G = S$. The stabilizer of any $x \in G$ is just the identity element:

$$\mathrm{Stab}(x) = \{g \in G \mid xg = x\} = \{e\}\,.$$

Therefore, the kernel of this action is $\mathrm{Ker}\,\phi = \bigcap_{x \in G} \mathrm{Stab}(x) = \{e\}$.

Since $\mathrm{Ker}\,\phi = \{e\}$, the homomorphism ϕ is injective. □

Example 5: Groups acting on themselves by conjugation

Another way a group G can act on itself (that is, $S = G$) is by **conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S), \qquad \phi(g) \text{ is the permutation that sends each } x \mapsto g^{-1}xg.$$

- The orbit of $x \in G$ is its conjugacy class:

$$\mathrm{Orb}(x) = \{x.\phi(g) \mid g \in G\} = \{g^{-1}xg \mid g \in G\} = \mathrm{cl}_G(x)\,.$$

- The stabilizer of x is the set of elements that commute with x; called its centralizer:

$$\mathrm{Stab}(x) = \{g \in G \mid g^{-1}xg = x\} = \{g \in G \mid xg = gx\} := C_G(x)$$

- The fixed points of ϕ are precisely those in the center of G:

$$\mathrm{Fix}(\phi) = \{x \in G \mid g^{-1}xg = x \text{ for all } g \in G\} = Z(G)\,.$$

By the Orbit-Stabilizer theorem, $|G| = |\,\mathrm{Orb}(x)| \cdot |\,\mathrm{Stab}(x)| = |\,\mathrm{cl}_G(x)| \cdot |C_G(x)|$. Thus, we immediately get the following new result about conjugacy classes:

Theorem 3

For any $x \in G$, the size of the conjugacy class $\mathrm{cl}_G(x)$ divides the size of G.

Example 5a: S_3 acting on itself by conjugation

Consider the action of $G = S_3$ on itself by **conjugation**.

The orbits of the action are the conjugacy classes: $\{e\}$, $\{$ permutations of the form $(ab)\}$, $\{$ permutations of the form $(abc)\}$

The fixed points of ϕ are elements of the size-1 conjugacy classes, only e has a size-one conjugacy classs.

By the Orbit-Stabilizer theorem:

$$|\operatorname{Stab}(x)| = \frac{|S_3|}{|\operatorname{Orb}(x)|} = \frac{6}{|\operatorname{cl}_G(x)|}.$$

The stabilizer subgroups are as follows:

- $\operatorname{Stab}(e) = S_3$,
- $\operatorname{Stab}(\ (123)\ ) = \operatorname{Stab}(\ (132)\ ) = \langle (123) \rangle \cong C_3$,
- $\operatorname{Stab}(\ (12)\ ) = \{e, (12)\} \cong C_2$
- $\operatorname{Stab}(\ (13)\ ) =$
- $\operatorname{Stab}(\ (23)\ ) =$

Example 5b: D_6 acting on itself by conjugation

Consider the action of $G = D_6$ on itself by **conjugation**.

The orbits of the action are the conjugacy classes:

e	r	r^2	f	r^2f	r^4f
r^3	r^5	r^4	rf	r^3f	r^5f

The fixed points of ϕ are the size-1 conjugacy classes. These are the elements in the center: $Z(D_6) = \{e\} \cup \{r^3\} = \langle r^3 \rangle$.

By the Orbit-Stabilizer theorem:

$$|\operatorname{Stab}(x)| = \frac{|D_6|}{|\operatorname{Orb}(x)|} = \frac{12}{|\operatorname{cl}_G(x)|}.$$

The stabilizer subgroups are as follows:

- $\operatorname{Stab}(e) = \operatorname{Stab}(r^3) = D_6$,
- $\operatorname{Stab}(r) = \operatorname{Stab}(r^2) = \operatorname{Stab}(r^4) = \operatorname{Stab}(r^5) = \langle r \rangle = C_6$,
- $\operatorname{Stab}(f) = \{e, r^3, f, r^3f\} = \langle r^3, f \rangle$,
- $\operatorname{Stab}(rf) = \{e, r^3, rf, r^4f\} = \langle r^3, rf \rangle$,
- $\operatorname{Stab}(r^if) = \{e, r^3, r^if, r^if\} = \langle r^3, r^if \rangle$.

(Not covered) Example 6: Groups acting on subgroups by conjugation

Let $G = D_3$, and let S be the set of proper <u>subgroups</u> of G:

$$S = \Big\{ \langle e \rangle,\ \langle r \rangle,\ \langle f \rangle,\ \langle rf \rangle,\ \langle r^2 f \rangle \Big\}.$$

There is a right group action of $D_3 = \langle r, f \rangle$ on S by conjugation:

$$\tau \colon D_3 \longrightarrow \mathrm{Perm}(S)\,, \qquad \tau(g) \text{ is the permutation that sends each } H \text{ to } g^{-1}Hg.$$

$\tau(e) \;=\; \langle e \rangle \quad \langle r \rangle \quad \langle f \rangle \quad \langle rf \rangle \quad \langle r^2 f \rangle$

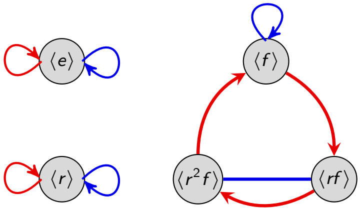
$\tau(r) \;=\; \langle e \rangle \quad \langle r \rangle \quad \langle f \rangle \quad \langle rf \rangle \quad \langle r^2 f \rangle$

$\tau(r^2) \;=\; \langle e \rangle \quad \langle r \rangle \quad \langle f \rangle \quad \langle rf \rangle \quad \langle r^2 f \rangle$

$\tau(f) \;=\; \langle e \rangle \quad \langle r \rangle \quad \langle f \rangle \quad \langle rf \rangle \quad \langle r^2 f \rangle$

$\tau(rf) \;=\; \langle e \rangle \quad \langle r \rangle \quad \langle f \rangle \quad \langle rf \rangle \quad \langle r^2 f \rangle$

$\tau(r^2 f) \;=\; \langle e \rangle \quad \langle r \rangle \quad \langle f \rangle \quad \langle rf \rangle \quad \langle r^2 f \rangle$



The action diagram.

$$\mathrm{Stab}(\langle e \rangle) = \mathrm{Stab}(\langle r \rangle) = D_3 = N_{D_3}(\langle r \rangle)$$

$$\mathrm{Stab}(\langle f \rangle) = \langle f \rangle = N_{D_3}(\langle f \rangle),$$

$$\mathrm{Stab}(\langle rf \rangle) = \langle rf \rangle = N_{D_3}(\langle rf \rangle),$$

$$\mathrm{Stab}(\langle r^2 f \rangle) = \langle r^2 f \rangle = N_{D_3}(\langle r^2 f \rangle).$$

(Not covered) Example 6: Groups acting on subgroups by conjugation

More generally, any group G acts on its set S of subgroups by **conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) \text{ is the permutation that sends each } H \text{ to } g^{-1}Hg.$$

This is a right action, but there is an associated left action: $H \mapsto gHg^{-1}$.

Let $H \leq G$ be an element of S.

- The orbit of H consists of all conjugate subgroups:

$$\mathrm{Orb}(H) = \{g^{-1}Hg \mid g \in G\}\,.$$

- The stabilizer of H is the normalizer of H in G:

$$\mathrm{Stab}(H) = \{g \in G \mid g^{-1}Hg = H\} = N_G(H)\,.$$

- The fixed points of ϕ are precisely the normal subgroups of G:

$$\mathrm{Fix}(\phi) = \{H \leq G \mid g^{-1}Hg = H \text{ for all } g \in G\}\,.$$

- The kernel of this action is G iff every subgroup of G is normal. In this case, ϕ is the trivial homomorphism: pressing the g-button fixes (i.e., normalizes) every subgroup.

Example 7: Groups acting on cosets of H by right-multiplication

Fix a subgroup $H \leq G$. Then G acts on its **right cosets** by **right-multiplication**:

$$\phi\colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Let Hx be an element of $S = G/H$ (the right cosets of H).

- There is only one orbit. For example, given two cosets Hx and Hy,

$$\phi(x^{-1}y) \text{ sends } Hx \longmapsto Hx(x^{-1}y) = Hy.$$

- The stabilizer of Hx is the conjugate subgroup $x^{-1}Hx$:

$$\text{Stab}(Hx) = \{g \in G \mid Hxg = Hx\} = \{g \in G \mid Hxgx^{-1} = H\} = x^{-1}Hx\,.$$

- Assuming $H \neq G$, there are no fixed points of ϕ. The only orbit has size $[G : H] > 1$.
- The kernel of this action is the intersection of all conjugate subgroups of H:

$$\text{Ker}\,\phi = \bigcap_{x \in G} x^{-1}Hx$$

Notice that $\langle e \rangle \leq \text{Ker}\,\phi \leq H$, and $\text{Ker}\,\phi = H$ iff $H \trianglelefteq G$.

Example 7a: S_3 acting on cosets of $H := \langle (12) \rangle$ by right multiplication