Math 3230 Abstract Algebra I
Sec 5.2: The orbit-stabilizer theorem

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Abstract Algebra I

Orbits, stabilizers, and fixed points

Suppose G acts on a set S. Pick a configuration $s \in S$. We can ask two questions about it:

(i) What other states (in S) are reachable from s? (We call this the orbit of s.)

(ii) What group elements (in G) fix s? (We call this the stabilizer of s.)

Definition 2

Suppose that G acts on a set S (on the right) via $\phi\colon G \to \mathrm{Perm}(S)$.

(i) The orbit of $s \in S$ is the set

$$\mathrm{Orb}(s) = \{s.\phi(g) \mid g \in G\}.$$

(ii) The stabilizer of s in G is

$$\mathrm{Stab}(s) = \{g \in G \mid s.\phi(g) = s\}.$$

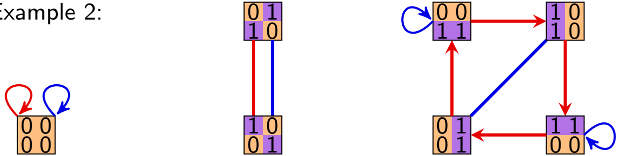
(iii) An element $s \in S$ is called a fixed point of the action if $\mathrm{Orb}(s)$ is of size one. That is, the set of fixed points of the action is

$$\mathrm{Fix}(\phi) = \{s \in S \mid s.\phi(g) = s \text{ for all } g \in G\}.$$

Note that the orbits of ϕ are the connected components in the action diagram.

Orbits, stabilizers, and fixed points

Let's revisit Example 2:



The orbits are the 3 connected components. There is only one fixed point of ϕ. The stabilizers are:

$$\mathrm{Stab}\left(\boxed{\begin{smallmatrix}0&0\\0&0\end{smallmatrix}}\right)=D_4,\qquad \mathrm{Stab}\left(\boxed{\begin{smallmatrix}0&1\\1&0\end{smallmatrix}}\right)=\{e,r^2,rf,r^3f\},\qquad \mathrm{Stab}\left(\boxed{\begin{smallmatrix}0&0\\1&1\end{smallmatrix}}\right)=\{e,f\},$$

$$\mathrm{Stab}\left(\boxed{\begin{smallmatrix}1&0\\0&1\end{smallmatrix}}\right)=\{e,r^2,rf,r^3f\},\qquad \mathrm{Stab}\left(\boxed{\begin{smallmatrix}1&0\\1&0\end{smallmatrix}}\right)=\{e,r^2f\},$$

$$\mathrm{Stab}\left(\boxed{\begin{smallmatrix}1&1\\0&0\end{smallmatrix}}\right)=\{e,f\},$$

Observations?

$$\mathrm{Stab}\left(\boxed{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}}\right)=\{e,r^2f\}.$$

Example 4: $G=C_2=\{e,g\}$ and $S=\mathbb{Z}$. Let G acts on S by $x.\phi(g)=-x$.

$\mathrm{Orb}(0)=\{0\}$, and the orbit of any other element x in S is the set $\{-x,x\}$.

$\mathrm{Stab}(0)=C_2$, but the stabilizer of any other element of S is $\{e\}$.

$\mathrm{Fix}(\phi)=\{0\}$.

Orbits and stabilizers

Proposition 1

For any $s \in S$, the set $\text{Stab}(s)$ is a subgroup of G.

Proof (outline) - see actual proof in video

To show $\text{Stab}(s)$ is a group, we need to show three things:

(i) *Contains the identity*. That is, $s.\phi(e) = s$.

(ii) *Inverses exist*. That is, if $s.\phi(g) = s$, then $s.\phi(g^{-1}) = s$.

(iii) *Closure*. That is, if $s.\phi(g) = s$ and $s.\phi(h) = s$, then $s.\phi(gh) = s$.

You'll do this on the homework.

Remark

The kernel of the action ϕ is the set of all group elements that fix everything in S:

$$\text{Ker}\,\phi = \{g \in G \mid \phi(g) = e\} = \{g \in G \mid s.\phi(g) = s \text{ for all } s \in S\}.$$

Notice that

$$\text{Ker}\,\phi = \bigcap_{s \in S} \text{Stab}(s).$$

Lemma 1: Bijection between orbits and right cosets of the stabilizer

Lemma 1

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and any $x \in S$, There is a bijection $f : \mathrm{Orb}(x) \to G/Stab(x)$.

Let's look at our previous example to get some intuition for why this should be true.

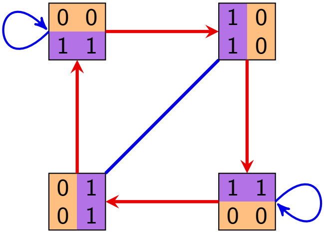
Let $x = \begin{array}{|cc|}\hline 0 & 0 \\ 1 & 1 \\ \hline\end{array}$

Then $\mathrm{Stab}(x) = \langle f \rangle$.

Partition of D_4 by the right cosets of H :

$G = D_4$ and $H = \langle f \rangle$

e	r	r^2	r^3
f	fr	fr^2	fr^3
H	Hr	Hr^2	Hr^3



Here, $x.\phi(g) = x.\phi(k)$ iff g and k are in the same right coset of H in G.

Theorem 1 (The Orbit-Stabilizer Theorem)

The following is a central result of group theory.

Orbit-Stabilizer theorem

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and any $x \in S$,

$$|\operatorname{Orb}(x)| \cdot |\operatorname{Stab}(x)| = |G|\,.$$

if G is finite.

Proof of Orbit-Stabilizer theorem

Since $\operatorname{Stab}(s) < G$, Lagrange's theorem tells us that

$$\underbrace{[G : \operatorname{Stab}(x)]}_{\text{number of cosets}} \cdot \underbrace{|\operatorname{Stab}(x)|}_{\text{size of subgroup}} = |G|.$$

Thus, it suffices to show that $|\operatorname{Orb}(x)| = [G : \operatorname{Stab}(x)]$.

By Lemma 1, there is a bijection between the elements of $\operatorname{Orb}(x)$ and the right cosets of $\operatorname{Stab}(x)$.

Proof of Lemma 1 part (i)

Lemma 1

For any group action $\phi \colon G \to \text{Perm}(S)$, and any $x \in S$, there is a bijection $f : \text{Orb}(x) \to G/\,\text{Stab}(x)$.

Proof of Lemma, (i) defining f and showing that f is well-defined

Throughout, let $H = \text{Stab}(x)$. We define $f : \text{Orb}(x) \to G/H$ as follows. If $s \in \text{Orb}(x)$, then there is $g \in G$ such that $s = x.\phi(g)$ by def of $\text{Orb}(x)$, and we define $f(s) = Hg$.

(i) First, we show that f is well-defined, independent of the choice of $g \in G$ such that $x.\phi(g) = s$. We need to show: *if two group elements both send x to $s \in S$, then the two group elements are in the same coset.*

Suppose $g, k \in G$ both send $x \in S$ to $s \in S$. This means:

$$\begin{aligned}
x.\phi(g) = x.\phi(k) \quad &\Rightarrow \quad (x.\phi(g)).\phi(k)^{-1} = (x.\phi(k)).\phi(k)^{-1} \\
&\Rightarrow \quad (x.\phi(g)).\phi(k^{-1}) = (x.\phi(k)).\phi(k^{-1}) \\
&\Rightarrow \quad x.\phi(gk^{-1}) = x.\phi(kk^{-1}) \qquad \text{since } \phi \text{ is a right group action} \\
&\Rightarrow \quad x.\phi(gk^{-1}) = x.\phi(e) \\
&\Rightarrow \quad x.\phi(gk^{-1}) = x \quad (\text{since } \phi \text{ sends } e \text{ to the identity permutation}) \\
&\Rightarrow \quad gk^{-1} \text{ stabilizes } x \\
&\Rightarrow \quad gk^{-1} \in H \quad (\text{recall that } H = \text{Stab}(x)) \\
&\Rightarrow \quad g \in Hk \\
&\Rightarrow \quad Hg = Hk
\end{aligned}$$

Proof of Lemma 1 part (ii) injectivity

Proof of Lemma (cont.), (ii) showing that f is injective

Suppose (a.) $x_1, x_2 \in \mathrm{Orb}(x)$ such that (b.) $f(x_1) = f(x_2)$. By (a), there exist $g_1, g_2 \in G$ such that

$$\begin{aligned} x_1 &= x.\phi(g_1) \\ x_2 &= x.\phi(g_2), \end{aligned}$$

with $Hg_1 = Hg_2$, due to (b). Then

$$hg_1 = g_2 \text{ for some } h \in H = \mathrm{Stab}(x). \tag{1}$$

So

$$\begin{aligned} x_2 &= x.\phi(g_2) \\ &= x.\phi(hg_1) && \text{by (1)} \\ &= (x.\phi(h)).\phi(g_1) && \text{since } \phi \text{ is a right group action} \\ &= x.\phi(g_1) && \text{since } x.\phi(h) = x \text{ due to } h \in H = \mathrm{Stab}(x) \\ &= x_1. \end{aligned}$$

Proof of Lemma 1 part (iii) surjectivity

Proof of Lemma (cont.), (iii) showing that f is onto

(iii) Finally, we show that f is onto.
Let $Hg_1 \in G/H$. Then let $x_1 := x.\phi(g_1)$, which means that $x_1 \in \mathrm{Orb}(x)$. Then $f(x_1) = Hg_1$.