Math 3230 Abstract Algebra I
Sec 4.2: Kernels

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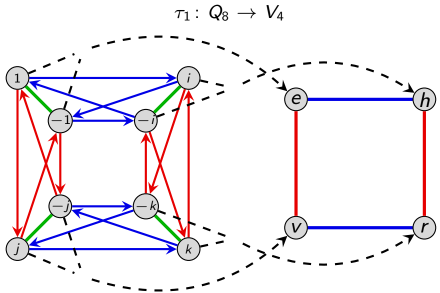
http://egunawan.github.io/algebra

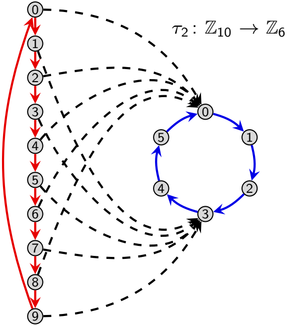
Abstract Algebra I

Quotient maps

Consider a homomorphism where more than one element of the domain maps to the same element of the codomain (i.e., non-embeddings).

Here are some examples.





Non-embedding homomorphisms are called quotient maps (as we'll see, they correspond to our quotient process).

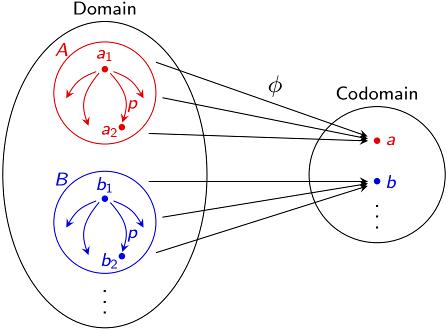
Preimages

Definition

If $\phi\colon G \to H$ is a homomorphism and $h \in \text{Im}(\phi) < H$, define the preimage (often called fiber) of h to be the set

$$\phi^{-1}(h) := \phi^{-1}(\{h\}) = \{g \in G : \phi(g) = h\}\,.$$

Observe in the previous examples that the preimages all had the same structure. This always happens.



The preimage of $1_H \in H$ is called the kernel of ϕ, denoted $\text{Ker}\,\phi$.

Preimages

Observation

All preimages of ϕ have the same structure.

Sketch of proof

Pick two elements $a, b \in \phi(G)$, and let $A = \phi^{-1}(a)$ and $B = \phi^{-1}(b)$ be their preimages.

Consider any path $a_1 \xrightarrow{p} a_2$ between elements in A. For any $b_1 \in B$, there is a corresponding path $b_1 \xrightarrow{p} b_2$. *We need to show that $b_2 \in B$.*

Since homomorphisms preserve structure, $\phi(a_1) \xrightarrow{\phi(p)} \phi(a_2)$. Since $\phi(a_1) = \phi(a_2)$, $\phi(p)$ is the *trivial path*.

Therefore, $\phi(b_1) \xrightarrow{\phi(p)} \phi(b_2)$, i.e., $\phi(b_1) = \phi(b_2)$, and so by definition, $b_2 \in B$. $\square$

Clearly, G is partitioned by preimages of ϕ. Additionally, we just showed that they all have the same structure. (Sound familiar?)

Preimages and kernels

Definition

The kernel of a homomorphism $\phi\colon G \to H$ is the set

$$\operatorname{Ker}(\phi) := \phi^{-1}(e) = \{k \in G : \phi(k) = e\}\,.$$

Recall

- The preimage of the identity (i.e., $K = \operatorname{Ker}(\phi)$) is a subgroup of G.

Proof

Let $K = \operatorname{Ker}(\phi)$, and take $a, b \in K$. We must show that K satisfies 3 properties:

Identity: $\phi(e) = e$. ✓

Closure: $\phi(ab) = \phi(a)\,\phi(b) = e \cdot e = e$. ✓

Inverses: $\phi(a^{-1}) = \phi(a)^{-1} = e^{-1} = e$. ✓

Thus, K is a subgroup of G. □

Note: All other preimages are left cosets of K.

A homomorphism is injective iff its kernel is trivial

Prop 2

A group homomorphism $\phi : G_1 \to G_2$ is injective if and only if $\mathrm{Ker}(\phi) = \{e_1\}$.

Proof:

Example 1 (Application of Prop 2): Determine all possible homomorphism $f : \mathbb{Z}_7 \to \mathbb{Z}_{12}$.

Answer: The only homomorphism in this case is the zero map. See Example 11.8 in Judson's textbook abstract.ups.edu/aata/section-group-homomorphisms.html

Example 2: The homomorphism from $GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by the matrix determinant. The kernel is the group $SL_2(\mathbb{R})$ of 2×2 matrices with determinant 1. See Example 11.6 in Judson's textbook abstract.ups.edu/aata/section-group-homomorphisms.html

Exercise: Show that det(AB) = det(A) det(B)

Example 3: If G is a group and $g \in G$, let $f : \mathbb{Z} \to G$ be the group homomorphism defined by $f(n) = g^n$. The kernel of f is trivial if the order of g is infinite. Otherwise, the kernel is $k\mathbb{Z}$ where k is the order of g.

Kernels are normal subgroups

Prop 3

If $\phi : G_1 \to G_2$ is a group homomorphism, then $\mathrm{Ker}(\phi)$ is a normal subgroup of G_1.

Proof

Let $K = \mathrm{Ker}(\phi)$. We will show that if $k \in K$, then $gkg^{-1} \in K$ for all $g \in G_1$. Take any $g \in G_1$, and observe that

$$\phi(gkg^{-1}) = \phi(g)\,\phi(k)\,\phi(g^{-1}) = \phi(g) \cdot e \cdot \phi(g^{-1}) = \phi(g)\phi(g)^{-1} = e\,.$$

Therefore, $gkg^{-1} \in \mathrm{Ker}(\phi)$, so $K \trianglelefteq G_1$. $\square$

See also Theorem 11.5 in Judson's textbook
abstract.ups.edu/aata/section-group-homomorphisms.html

IMPORTANT OBSERVATION!

Given any homomorphism $\phi\colon G_1 \to G_2$, we can *always* form the quotient group

$$G_1/\,\mathrm{Ker}(\phi).$$

Example 4. Quotients: via multiplication tables

Recall that $C_2 = \{e^{0\pi i}, e^{1\pi i}\} = \{1, -1\}$. Consider the following (quotient) homomorphism:

$$\phi\colon D_4 \longrightarrow C_2\,, \qquad \text{defined by } \phi(r) = 1 \text{ and } \phi(f) = -1\,.$$

Note that $\phi(\text{rotation}) = 1$ and $\phi(\text{reflection}) = -1$.

The quotient process of "shrinking D_4 down to C_2" can be clearly seen from the multiplication tables.

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

non-flip **flip**

flip **non-flip**

	1	-1
1	1	-1
-1	-1	1

Example 5. Quotients: via Cayley diagrams

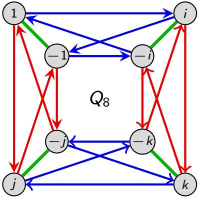
Define the homomorphism $\phi : Q_8 \to V_4$ via $\phi(i) = v$ and $\phi(j) = h$. Since $Q_8 = \langle i, j \rangle$, we can determine where ϕ sends the remaining elements:

$$\phi(1) = e\,, \qquad \phi(-1) = \phi(i^2) = \phi(i)^2 = v^2 = e\,,$$

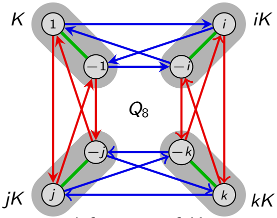
$$\phi(k) = \phi(ij) = \phi(i)\phi(j) = vh = r\,, \qquad \phi(-k) = \phi(ji) = \phi(j)\phi(i) = hv = r\,,$$

$$\phi(-i) = \phi(-1)\phi(i) = ev = v\,, \qquad \phi(-j) = \phi(-1)\phi(j) = eh = h\,.$$

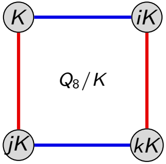
Note that $\mathrm{Ker}\,\phi = \{-1, 1\}$. Let's see what happens when we quotient out by $\mathrm{Ker}\,\phi$:



Q_8 organized by the subgroup $K = \langle -1 \rangle$



left cosets of K are near each other



collapse cosets into single nodes

Do you notice any relationship between $Q_8/\,\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$?