Math 3230 Abstract Algebra I
Sec 3.7: Conjugacy classes

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Abstract Algebra I

Conjugation

Recall that for $H \leq G$, the conjugate subgroup of H by a fixed $g \in G$ is

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}.$$

Additionally, H is normal iff $gHg^{-1} = H$ for all $g \in G$.

We can also fix the element we are conjugating. Given $x \in G$, we may ask:

"*which elements can be written as gxg^{-1} for some $g \in G$?*"

The set of all such elements in G is called the conjugacy class of x, denoted $\text{cl}_G(x)$. Formally, this is the set

$$\text{cl}_G(x) = \{gxg^{-1} \mid g \in G\}.$$

Remarks

- In any group, $\text{cl}_G(e) = \{e\}$, because $geg^{-1} = e$ for any $g \in G$.
- If x and g commute, then $gxg^{-1} = x$. Thus, when computing $\text{cl}_G(x)$, we only need to check gxg^{-1} for those $g \in G$ that *do not commute* with x.
- Moreover, $\text{cl}_G(x) = \{x\}$ iff x commutes with everything in G. (Why?)

Conjugacy classes

Proposition 1

Conjugacy is an equivalence relation.

Proof

- *Reflexive*: $x = exe^{-1}$.
- *Symmetric*: $x = gyg^{-1} \Rightarrow y = g^{-1}xg$.
- *Transitive*: $x = gyg^{-1}$ and $y = hzh^{-1} \Rightarrow x = (gh)z(gh)^{-1}$. $\square$

Since conjugacy is an equivalence relation, it partitions the group G into equivalence classes (conjugacy classes).

Let's compute the conjugacy classes in D_4. We'll start by finding $\mathrm{cl}_{D_4}(r)$. Note that we only need to compute grg^{-1} for those g that *do not* commute with r:

$$frf^{-1} = r^3, \qquad (rf)r(rf)^{-1} = r^3, \qquad (r^2f)r(r^2f)^{-1} = r^3, \qquad (r^3f)r(r^3f)^{-1} = r^3.$$

Therefore, the conjugacy class of r is $\mathrm{cl}_{D_4}(r) = \{r, r^3\}$.

Since conjugacy is an equivalence relation, $\mathrm{cl}_{D_4}(r^3) = \mathrm{cl}_{D_4}(r) = \{r, r^3\}$.

Conjugacy classes in D_4

To compute $\text{cl}_{D_4}(f)$, we don't need to check e, r^2, f, or $r^2 f$, since these all commute with f:

$$rfr^{-1} = r^2 f, \qquad r^3 f (r^3)^{-1} = r^2 f, \qquad (rf)f(rf)^{-1} = r^2 f, \qquad (r^3 f)f(r^3 f)^{-1} = r^2 f\,.$$

Therefore, $\text{cl}_{D_4}(f) = \{f, r^2 f\}$.

What is $\text{cl}_{D_4}(rf)$? Note that it has size greater than 1 because rf does not commute with everything in D_4.

It also *cannot* contain elements from the other conjugacy classes. The only element left is $r^3 f$, so $\text{cl}_{D_4}(rf) = \{rf, r^3 f\}$.

The "Class Equation", visually:
Partition of D_4 by its
conjugacy classes

e	r	f $\quad r^2 f$
r^2	r^3	rf $\quad r^3 f$

We can write $D_4 = \underbrace{\{e\} \cup \{r^2\}}_{\text{these commute with everything in } D_4} \cup \{r, r^3\} \cup \{f, r^2 f\} \cup \{r, r^3 f\}$.

The class equation

Definition

The center of G is the set $Z(G) = \{z \in G \mid gz = zg, \;\; \forall g \in G\}$.

Observation

$\mathrm{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.

Proof

Suppose x is its own conjugacy class. This means that

$$\mathrm{cl}_G(x) = \{x\} \iff gxg^{-1} = x, \;\; \forall g \in G \iff gx = xg, \;\; \forall g \in G \iff x \in Z(G).$$

$\square$

The Class Equation

For any finite group G,

$$|G| = |Z(G)| + \sum |\,\mathrm{cl}_G(x_i)|$$

where the sum is taken over distinct conjugacy classes of size greater than 1.

More on conjugacy classes

Proposition 2

Every normal subgroup is the union of conjugacy classes.

Proof

Suppose $n \in N \trianglelefteq G$. Then $gng^{-1} \in gNg^{-1} = N$, thus if $n \in N$, its entire conjugacy class $\mathrm{cl}_G(n)$ is contained in N as well. $\square$

Proposition 3

Conjugate elements have the same order.

Proof

Consider x and $y = gxg^{-1}$.

If $x^n = e$, then $(gxg^{-1})^n = (gxg^{-1})(gxg^{-1}) \cdots (gxg^{-1}) = gx^n g^{-1} = geg^{-1} = e$. Therefore, $|x| \geq |gxg^{-1}|$.

Conversely, if $(gxg^{-1})^n = e$, then $gx^n g^{-1} = e$, and it must follow that $x^n = e$. Therefore, $|x| \leq |gxg^{-1}|$. $\square$

Conjugacy classes in D_6

Let's determine the conjugacy classes of $D_6 = \langle r, f \mid r^6 = e,\ f^2 = e,\ r^i f = f r^{-i} \rangle$.

The center of D_6 is $Z(D_6) = \{e, r^3\}$; these are the *only* elements in size-1 conjugacy classes.

The only two elements of order 6 are r and r^5; so we must have $\mathrm{cl}_{D_6}(r) = \{r, r^5\}$.

The only two elements of order 3 are r^2 and r^4; so we must have $\mathrm{cl}_{D_6}(r^2) = \{r^2, r^4\}$.

Let's compute the conjugacy class of a reflection $r^i f$. We need to consider two cases; conjugating by r^j and by $r^j f$:

- $r^j (r^i f) r^{-j} = r^j r^i r^j f = r^{i+2j} f$
- $(r^j f)(r^i f)(r^j f)^{-1} = (r^j f)(r^i f) f r^{-j} = r^j f r^{i-j} = r^j r^{j-i} f = r^{2j-i} f$.

Thus, $r^i f$ and $r^k f$ are conjugate iff i and k are both even, or both odd.

The Class Equation, visually:
Partition of D_6 by its conjugacy classes

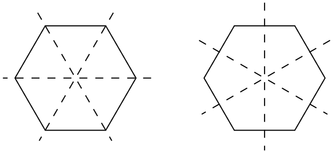
e	r	r^2	f	$r^2 f$	$r^4 f$
r^3	r^5	r^4	rf	$r^3 f$	$r^5 f$

Conjugacy "preserves structure"

Think back to linear algebra. Two matrices A and B are *similar* (=conjugate) if $A = PBP^{-1}$.

Conjugate matrices have the same eigenvalues, eigenvectors, and determinant. In fact, they represent the *same linear map*, but under a change of basis.

If n is even, then there are two "types" of reflections of an n-gon: the axis goes through two corners, or it bisects a pair of sides.



Notice how in D_n, conjugate reflections have the same "type." Do you have a guess of what the conjugacy classes of reflections are in D_n when n is odd?

Also, conjugate rotations in D_n had the same rotating angle, but in the opposite direction (e.g., r^k and r^{n-k}).

Next, we will look at conjugacy classes in the symmetric group S_n. We will see that conjugate permutations have "the same structure."

Cycle type and conjugacy

Definition

Two elements in S_n have the same cycle type if when written as a product of disjoint cycles, there are the same number of length-k cycles for each k.

We can write the cycle type of a permutation $\sigma \in S_n$ as a list $c_1, c_2, \ldots, c_n$, where c_i is the number of cycles of length i in σ.
Here is an example of some elements in S_9 and their cycle types.

- $(1\ 8)\,(5)\,(2\ 3)\,(4\ 9\ 6\ 7)$ has cycle type 1,2,0,1.
- $(1\ 8\ 4\ 2\ 3\ 4\ 9\ 6\ 7)$ has cycle type 0,0,0,0,0,0,0,0,1.
- $e = (1)(2)(3)(4)(5)(6)(7)(8)(9)$ has cycle type 9.

Lemma 4 (One of the questions from HW3)

For any $\sigma \in S_n$, $\sigma^{-1}(a_1\ a_2\ \ldots\ a_k)\sigma = (\sigma(a_1)\ \sigma(a_2)\ \ldots\ \sigma(a_k))$

This means that two k-cycles are conjugate!

Exercise: Show that $x = (12), y = (14) \in S_6$ are conjugate by finding a permutation $\sigma \in S_6$ such that $sigma^{-1} x\sigma = y$.

Cycle type and conjugacy

Theorem

Two elements $g, h \in S_n$ are conjugate if and only if they have the same cycle type.

Proof:

Big idea

Conjugate permutations have the same structure. Such permutations are *the same up to renumbering*.

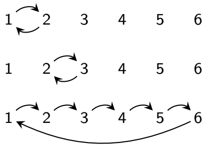
An example

Consider the following permutations in $G = S_6$:

$$g = (1\ 2)$$

$$h = (2\ 3)$$

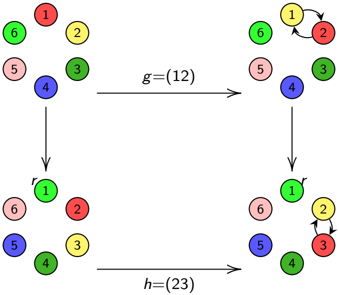
$$r = (1\ 2\ 3\ 4\ 5\ 6)$$



Since g and h have the same cycle type, they are conjugate:

$$(1\ 2\ 3\ 4\ 5\ 6)\,(2\ 3)\,(1\ 6\ 5\ 4\ 3\ 2) = (1\ 2)\,.$$

Here is a visual interpretation of $g = rhr^{-1}$:



Conjugacy: elements vs. groups

Remark

We can conjugate elements, or we can conjugate subgroups.

Conjugating elements defines an equivalence class on G.

- The equivalence classes have a special name (conjugacy classes) and notation, $\mathrm{cl}_G(x) = \{gxg^{-1} \mid g \in G\}$.
- Conjugate elements "have the same structure"; in particular, the same order.
- An element z has a unique conjugate iff $z \in Z(G)$, i.e., z commutes with everything.

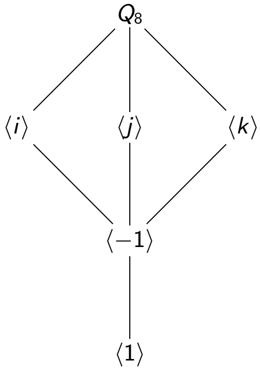
Conjugating subgroups defines an equivalence class on the set of subgroups of G.

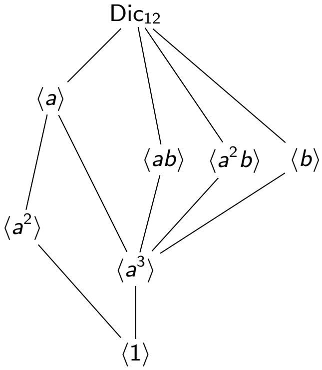
- The equivalence classes have a no special name or notation; we just call them conjugate subgroups to H, and write $\{xHx^{-1} \mid x \in G\}$.
- Conjugate subgroups have the same structure: they're isomorphic.
- A subgroup N has a unique conjugate iff N is normal.

In Section 5 (group actions), we'll learn more about the structure of these equivalence classes, such as how many there can be, and their possible sizes.

Conjugate subgroups

An an exercise, let's try to partition the following subgroup lattices into (conjugacy) equivalence classes.





$$Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$$

$$\begin{aligned} \mathrm{Dic}_{12} &= \langle a, b, c \mid a^3 = b^2 = c^2 = abc \rangle \\ &= \langle a, b \mid a^4 = b^3 = 1,\ bab = a \rangle \end{aligned}$$