Math 3230 Abstract Algebra I
Sec 3.6: Normalizers

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http://egunawan.github.io/algebra

Abstract Algebra I

Normalizers

Question

If $H \leq G$ but H is *not* normal, can we measure "how far" H is from being normal?

Recall that $H \trianglelefteq G$ iff $gH = Hg$ for all $g \in G$. So, one way to answer our question is to check how many $g \in G$ satisfy this requirement. Imagine that each $g \in G$ is voting as to whether H is normal:

$gH = Hg$ "yea" $\qquad\qquad$ $gH \neq Hg$ "nay"

At a *minimum*, every $g \in H$ votes "yea." (Why?)

At a *maximum*, every $g \in G$ could vote "yea," but this only happens when H really is normal.

There can be levels between these 2 extremes as well.

Definition

The set of elements in G that vote in favor of H's normality is called the normalizer of H in G, denoted $N_G(H)$. That is,

$$N_G(H) = \{g \in G : gH = Hg\} = \{g \in G : gHg^{-1} = H\}.$$

Normalizers

Let's explore some possibilities for what the normalizer of a subgroup can be. In particular, is it a subgroup?

Observation 1

If $g \in N_G(H)$, then $gH \subseteq N_G(H)$.

Proof

If $gH = Hg$, then $gH = bH$ for all $b \in gH$. Therefore, $bH = gH = Hg = Hb$. $\square$

The deciding factor in how a left coset votes is whether it is a right coset (members of gH vote as a block – exactly when $gH = Hg$).

Observation 2

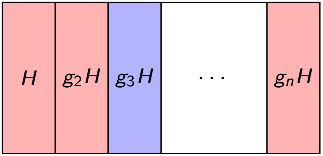
$|N_G(H)|$ is a multiple of $|H|$.

Proof

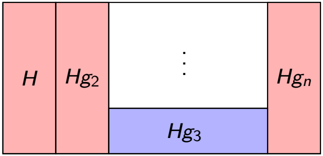
By Observation 1, $N_G(H)$ is made up of whole (left) cosets of H, and all (left) cosets are the same size and disjoint. $\square$

Normalizers

Consider a subgroup $H \leq G$ of index n. Suppose that the left and right cosets partition G as shown below:



Partition of G by the left cosets of H



Partition of G by the right cosets of H

The cosets H, and $g_2H = Hg_2$, and $g_nH = Hg_n$ all vote "yea".

The left coset g_3H votes "nay" because $g_3H \neq Hg_3$.

Assuming all other cosets vote "nay", the normalizer of H is

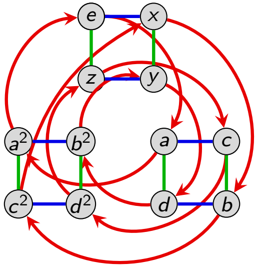
$$N_G(H) = H \cup g_2H \cup g_nH\,.$$

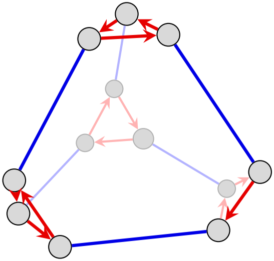
In summary, the two "extreme cases" for $N_G(H)$ are:

- $N_G(H) = G$: iff H is a normal subgroup
- $N_G(H) = H$: H is as "unnormal as possible"

An example: A_4

We saw earlier that $H = \langle x, z \rangle \trianglelefteq A_4$. Therefore, $N_{A_4}(H) = A_4$.





At the other extreme, consider $\langle a \rangle < A_4$ again, which is as far from normal as it can possibly be: $\langle a \rangle \ntrianglelefteq A_4$.

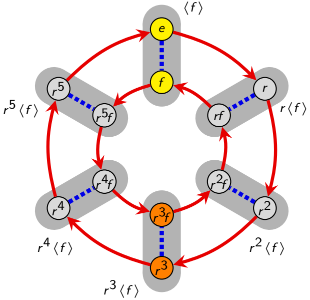
No right coset of $\langle a \rangle$ coincides with a left coset, other than $\langle a \rangle$ itself. Thus, $N_{A_4}(\langle a \rangle) = \langle a \rangle$.

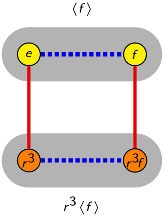
Observation 3

In the Cayley diagram of G, the normalizer of H consists of the copies of H that are connected to H by unanimous arrows.

How to spot the normalizer in the Cayley diagram

The following figure depicts the six left cosets of $H = \langle f \rangle = \{e, f\}$ in D_6.





Note that r^3H is the *only* coset of H (besides H, obviously) that cannot be reached from H by more than one element of D_6.

Thus, $N_{D_6}(\langle f \rangle) = \langle f \rangle \cup r^3\langle f \rangle = \{e, f, r^3, r^3 f\} \cong V_4$.

Observe that the normalizer is also a subgroup satisfying: $\langle f \rangle \lneq N_{D_6}(\langle f \rangle) \lneq D_6$.

Do you see the pattern for $N_{D_n}(\langle f \rangle)$? (It depends on whether n is even or odd.)

Normalizers are subgroups!

Theorem

For any $H \leq G$, we have $N_G(H) \leq G$.

Proof

Recall that $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$; "*the set of elements that normalize H.*" We need to verify three properties of $N_G(H)$:

(i) Contains the identity;

(ii) Inverses exist;

(iii) Closed under the binary operation.

Identity. Naturally, $eHe^{-1} = \{ehe^{-1} \mid h \in H\} = H$.

Inverses. Suppose $g \in N_G(H)$, which means $gHg^{-1} = H$. We need to show that $g^{-1} \in N_G(H)$. That is, $g^{-1}H(g^{-1})^{-1} = g^{-1}Hg = H$. Indeed,

$$g^{-1}Hg = g^{-1}(gHg^{-1})g = eHe = H\,.$$

Normalizers are subgroups!

Proof (cont.)

Closure. Suppose $g_1, g_2 \in N_G(H)$, which means that $g_1 H g_1^{-1} = H$ and $g_2 H g_2^{-1} = H$. We need to show that $g_1 g_2 \in N_G(H)$.

$$(g_1 g_2) H (g_1 g_2)^{-1} = g_1 g_2 H g_2^{-1} g_1^{-1} = g_1 (g_2 H g_2^{-1}) g_1^{-1} = g_1 H g_1^{-1} = H\,.$$

Since $N_G(H)$ contains the identity, every element has an inverse, and is closed under the binary operation, it is a (sub)group! □

Corollary

Every subgroup is normal in its normalizer:

$$H \trianglelefteq N_G(H) \leq G\,.$$

Proof

By definition, $gH = Hg$ for all $g \in N_G(H)$. Therefore, $H \trianglelefteq N_G(H)$. □