Math 3230 Abstract Algebra I
Sec 3.5: Quotients

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Abstract Algebra I

Quotients

Direct products make larger groups from smaller groups. It is a way to *multiply* groups.

The opposite procedure is called taking a quotient. It is a way to *divide* groups.

Unlike what we did with direct products, we will first describe the quotient operation using Cayley diagrams, and then formalize it algebraically and explore properties of the resulting group.

Definition

To divide a group G by one of its subgroups H, follow these steps:

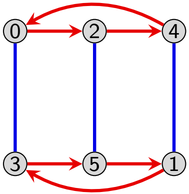
1. Organize a Cayley diagram of G by H (so that we can "see" the subgroup H in the diagram for G).
2. Collapse each left coset of H into one large node. Unite those arrows that now have the same start and end nodes. This forms a new diagram with fewer nodes and arrows.
3. IF (and *only* if) the resulting diagram is a Cayley diagram of a group, you have obtained the quotient group of G by H, denoted G/H (say: "G mod H".) If not, then G cannot be divided by H.

An example: $\mathbb{Z}_3 < \mathbb{Z}_6$

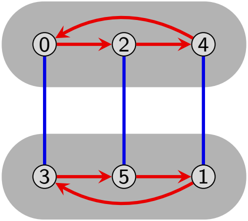
Consider the group $G = \mathbb{Z}_6$ and its normal subgroup $H = \langle 2 \rangle = \{0, 2, 4\}$.

There are two (left) cosets: $H = \{0, 2, 4\}$ and $1 + H = \{1, 3, 5\}$.

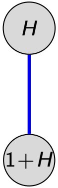
The following diagram shows how to take a quotient of $\mathbb{Z}_6$ by H.



$\mathbb{Z}_6$ organized by the subgroup $H = \langle 2 \rangle$



Left cosets of H are near each other



Collapse cosets into single nodes

In this example, the resulting diagram *is* a Cayley diagram. So, we *can* divide $\mathbb{Z}_6$ by $\langle 2 \rangle$, and we see that $\mathbb{Z}_6/H$ is isomorphic to $\mathbb{Z}_2$.

We write this as $\mathbb{Z}_6/H \cong \mathbb{Z}_2$.

A few remarks

- Step 3 of the Definition says "IF the new diagram is a Cayley diagram ..." Sometimes it won't be, in which case there is no quotient.
- The elements of G/H are the cosets of H. Asking if G/H exists amounts to asking if the set of left (or right) cosets of H forms a group. (More on this later.)
- In light of this, given *any* subgroup $H < G$ (regardless of whether the quotient process works), we will let

  $$G/H := \{gH \mid g \in G\}$$

  denote the set of left cosets of H in G.
- Not surprisingly, if $G = A \times B$ and we divide G by A (technically $A \times \{e\}$), the quotient group is B. (We'll see why shortly).

Caveat!

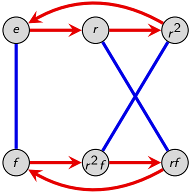
The converse of the previous statement is generally *not* true. That is, if G/H is a group, then G is in general *not* a direct product of H and G/H.

An example: $C_3 < D_3$

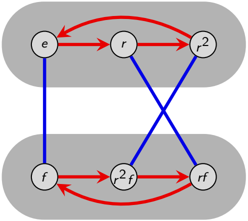
Consider the group $G = D_3$ and its normal subgroup $H = \langle r \rangle \cong C_3$.

There are two (left) cosets: $H = \{e, r, r^2\}$ and $fH = \{f, rf, r^2f\}$.

The following diagram shows how to take a quotient of D_3 by H.



D_3 organized by the subgroup $H = \langle r \rangle$



Left cosets of H are near each other

Collapse cosets into single nodes

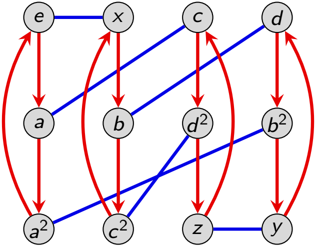
The result is a Cayley diagram for C_2, thus

$$D_3/H \cong C_2\,. \qquad \text{However}\ldots \qquad C_3 \times C_2 \ncong D_3\,.$$

Note that $C_3 \times C_2$ is abelian, but D_3 is not.

Example: $G = A_4$ and $H = \langle x, z \rangle \cong V_4$

Consider the following Cayley diagram for $G = A_4$ using generators $\langle a, x \rangle$.

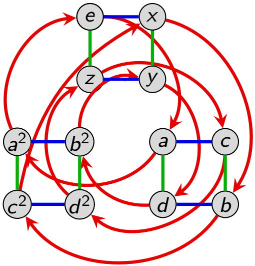


Consider $H = \langle x, z \rangle = \{e, x, y, z\} \cong V_4$. This subgroup is not "visually obvious" in this Cayley diagram.

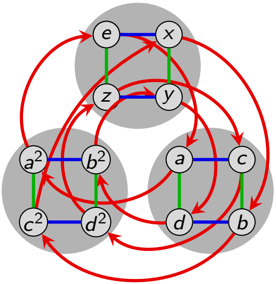
Let's add z to the generating set, and consider the resulting Cayley diagram.

Example: $G = A_4$ and $H = \langle x, z \rangle \cong V_4$

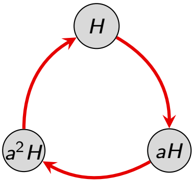
Here is a Cayley diagram for A_4 (with generators x, z, and a), organized by the subgroup $H = \langle x, z \rangle$ which allows us to see the left cosets of H clearly.



A_4 organized by the subgroup $H = \langle x, z \rangle$



Left cosets of H are near each other

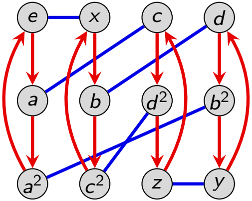


Collapse cosets into single nodes

The resulting diagram is a Cayley diagram! Therefore, $A_4/H \cong C_3$. However, A_4 is *not* isomorphic to the (abelian) group $V_4 \times C_3$.

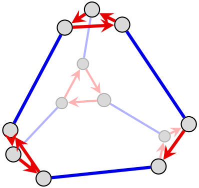
Example: $G = A_4$ and $H = \langle a \rangle \cong C_3$

Let's see an example where we cannot divide G by a particular subgroup H.

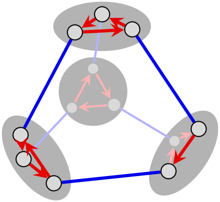


Consider the subgroup $H = \langle a \rangle \cong C_3$ of A_4.

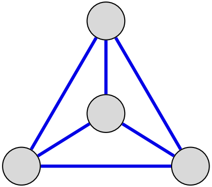
Do you see what will go wrong if we try to divide A_4 by $H = \langle a \rangle$?



A_4 organized by the subgroup $H = \langle a \rangle$



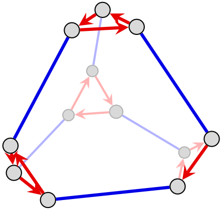
Left cosets of H are near each other



Collapse cosets into single nodes

This resulting diagram is *not* a Cayley diagram! There are multiple outgoing blue arrows from each node.

When can we divide G by a subgroup H?



Consider $H = \langle a \rangle \leq A_4$ again.

The left cosets are easy to spot.

Remark

The right cosets are *not* the same as the left cosets! The blue arrows out of any single coset scatter the nodes.

Thus, $H = \langle a \rangle$ is *not* normal in A_4.

If we took the effort to check our first 3 examples, we would find that in each case, the left cosets and right cosets coincide. In those examples, G/H existed, and H was normal in G.

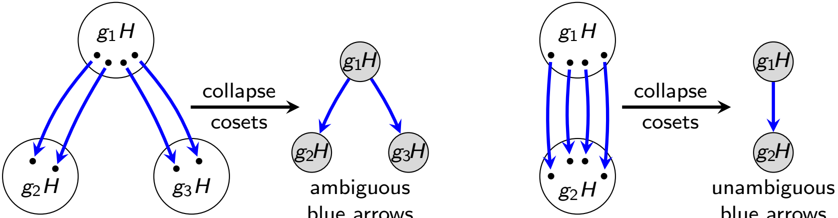
However, these 4 examples do not constitute a proof; they only provide evidence that the claim is true.

When can we divide G by a subgroup H?

Let's try to gain more insight. Consider a group G with subgroup H. Recall that:

- each **left coset** gH is the set of nodes that the H-arrows can reach from g (which looks like a copy of H at g);
- each **right coset** Hg is the set of nodes that the g-arrows can reach from H.

The following figure depicts the potential ambiguity that may arise when cosets are collapsed in the sense of our quotient definition.



blue arrows go from g_1H to multiple left cosets

blue arrows go from g_1H to a unique left coset

The action of the blue arrows above illustrates multiplication of a left coset on the right by some element. That is, the picture shows how left and right cosets *interact*.

When can we divide G by a subgroup H?

When H is normal, $gH = Hg$ for all $g \in G$.

In this case, to whichever coset one g arrow leads from H (the left coset), *all* g *arrows lead unanimously and unambiguously* (because it is also a right coset Hg).

Thus, in this case, collapsing the cosets is a *well-defined* operation.

Finally, we have an answer to our original question of when we can take a quotient.

Quotient theorem

If $H \leq G$, then the quotient group G/H can be constructed *if and only if* $H \trianglelefteq G$.

To summarize our "visual argument": The quotient process succeeds iff the resulting diagram is a valid Cayley diagram.

Nearly all aspects of valid Cayley diagrams are guaranteed by the quotient process: Every node has exactly one incoming and outgoing edge of each color, because $H \trianglelefteq G$. The diagram is regular too.

Though it's convincing, this argument isn't quite a formal proof; we'll do a rigorous algebraic proof next.

Quotient groups, algebraically

To prove the Quotient Theorem, we need to describe the quotient process algebraically.

Recall that even if H is not normal in G, we will still denote the set of left cosets of H in G by G/H.

Quotient theorem (restated)

When $H \trianglelefteq G$, the set of cosets G/H forms a group.

This means there is a well-defined binary operation on the set of cosets. But how do we "multipy" two cosets?

If aH and bH are left cosets, define

$$aH \cdot bH := abH\,.$$

Clearly, G/H is closed under this operation. But we also need to verify that this definition is well-defined.

By this, we mean that it *does not depend* on our choice of coset representative.

Quotient groups, algebraically

Lemma

Let $H \trianglelefteq G$. Multiplication of cosets is well-defined:

$$\text{if } a_1H = a_2H \text{ and } b_1H = b_2H, \text{ then } a_1H \cdot b_1H = a_2H \cdot b_2H.$$

Proof

Suppose that $H \trianglelefteq G$, $a_1H = a_2H$ and $b_1H = b_2H$. Then

$$\begin{aligned}
a_1H \cdot b_1H &= a_1b_1H && \text{(by definition)} \\
&= a_1(b_2H) && (b_1H = b_2H \text{ by assumption}) \\
&= (a_1H)b_2 && (b_2H = Hb_2 \text{ since } H \trianglelefteq G) \\
&= (a_2H)b_2 && (a_1H = a_2H \text{ by assumption}) \\
&= a_2b_2H && (b_2H = Hb_2 \text{ since } H \trianglelefteq G) \\
&= a_2H \cdot b_2H && \text{(by definition)}
\end{aligned}$$

Thus, the binary operation on G/H is well-defined. $\square$

Quotient groups, algebraically

Quotient theorem (restated)

When $H \trianglelefteq G$, the set of cosets G/H forms a group.

Proof

There is a well-defined binary operation on the set of left (equivalently, right) cosets: $aH \cdot bH = abH$. We need to verify the three remaining properties of a group:

Identity. The coset $H = eH$ is the identity because for any coset $aH \in G/H$,

$$aH \cdot H = aeH = aH = eaH = H \cdot aH\,.$$

Inverses. Given a coset aH, its inverse is $a^{-1}H$, because

$$aH \cdot a^{-1}H = eH = a^{-1}H \cdot aH\,.$$

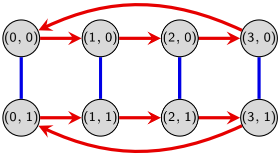
Closure. This is immediate, because $aH \cdot bH = abH$ is another coset in G/H. $\square$

Properties of quotients

Question

If H and K are subgroups and $H \cong K$, then are G/H and G/K isomorphic?

For example, here is a Cayley diagram for the group $\mathbb{Z}_4 \times \mathbb{Z}_2$:



It is visually obvious that the quotient of $\mathbb{Z}_4 \times \mathbb{Z}_2$ by the subgroup $\langle (0,1) \rangle \cong \mathbb{Z}_2$ is the group $\mathbb{Z}_4$.

The quotient of $\mathbb{Z}_4 \times \mathbb{Z}_2$ by the subgroup $\langle (2,0) \rangle \cong \mathbb{Z}_2$ is a bit harder to see. Algebraically, it consists of the cosets

$$\langle (2,0) \rangle\,, \qquad (1,0) + \langle (2,0) \rangle\,, \qquad (0,1) + \langle (2,0) \rangle\,, \qquad (1,1) + \langle (2,0) \rangle\,.$$

It is now apparent that this group is isomorphic to V_4.

Thus, the answer to the question above is "no." Surprised?