Direct products

If A and B are groups, there is a natural group structure on the set

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

Definition

The **direct product** of groups A and B consists of the set $A \times B$, and the group operation is done componentwise: if $(a, b), (c, d) \in A \times B$, then

$$(a, b) * (c, d) = (ac, bd).$$

We call A and B the factors of the direct product.

Note that the binary operations on A and B could be different. One might be $*$ and the other $+$.

For example, in $D_3 \times \mathbb{Z}_4$:

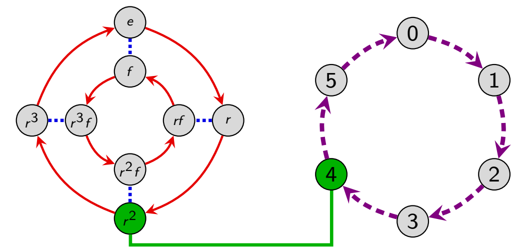
$$(r^2, 1) * (fr, 3) = (r^2 fr, 1 + 3) = (rf, 0).$$

These elements do *not* commute:

$$(fr, 3) * (r^2, 1) = (fr^3, 3 + 1) = (f, 0).$$

Direct products, visually

- One way to think of the direct product of two cyclic groups, say $\mathbb{Z}/n \times \mathbb{Z}/m$: Imagine a slot machine with two wheels, one with spaces numbered 0 through $n-1$, and the other with spaces numbered 0 through $m-1$.
  The actions are: spin both of wheels. Each action can be labeled by where we end up on each wheel, say (i, j).
- An example for a more *general* case: the element $(r^2, 4)$ in $D_4 \times \mathbb{Z}/6$.



Key idea

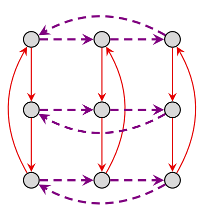
The direct product of two groups joins them so they act independently of each other.

Cayley diagrams of direct products

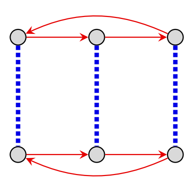
Remark

Just because a group is not written with $\times$ doesn't mean it isn't secretly a direct product. Example: V_4 (rectangle puzzle) is really $C_2 \times C_2$ (two light switches).

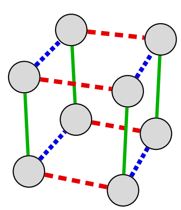
Here are some examples of direct products:



$C_3 \times C_3$



$C_3 \times C_2$



$C_2 \times C_2 \times C_2$

$C_2 \times C_2 \times C_2 \times C_2$

Surprisingly, the group $C_3 \times C_2$ is isomorphic to the cyclic group C_6! The Cayley diagram for C_6 using generators r^2 and r^3 is the same as the Cayley diagram for $C_3 \times C_2$ above.

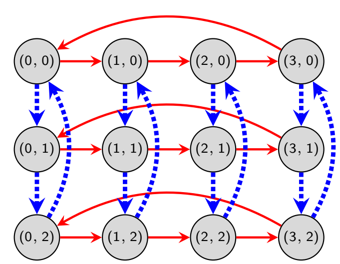
Cayley diagrams of direct products

Let e_A be the identity of A and e_B the identity of B.

Given a Cayley diagram of A with generators $a_1, \dots, a_k$, and a Cayley diagram of B with generators $b_1, \dots, b_\ell$, we can create a Cayley diagram for $A \times B$ as follows:

- Vertex set: $\{(a, b) \mid a \in A,\ b \in B\}$.
- Generators: $(a_1, e_b), \dots, (a_k, e_b)$ and $(e_a, b_1), \dots, (e_a, b_\ell)$.

It is helpful to arrange the vertices in a rectangular grid. For example, here is a Cayley diagram for the group $\mathbb{Z}/4 \times \mathbb{Z}/3$:



Exercise: List all subgroups of $\mathbb{Z}/4 \times \mathbb{Z}/3$.
Hint: There are six.

Subgroups of direct products

Proposition

If $H \leq A$, and $K \leq B$, then $H \times K$ is a subgroup of $A \times B$.

For $\mathbb{Z}/4 \times \mathbb{Z}/3$, *all* subgroups had this form. However, this is false in general!

Example when this is not true: Consider the group $\mathbb{Z}/2 \times \mathbb{Z}/2$, which is also V_4. Since $\mathbb{Z}/2$ has two subgroups, the following four sets are subgroups of $\mathbb{Z}/2 \times \mathbb{Z}/2$:

$$\mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \{0\} \times \{0\}, \qquad \mathbb{Z}_2 \times \{0\} = \langle (1,0) \rangle, \qquad \{0\} \times \mathbb{Z}_2 = \langle (0,1) \rangle.$$

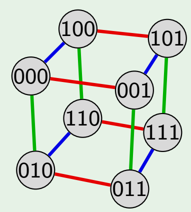
One subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$ is missing from this list: $\langle (1,1) \rangle = \{(0,0),(1,1)\}$.

Practice

What are the subgroups of $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$?

Here is a Cayley diagram, writing the elements of the product as abc rather than (a,b,c).

Hint: There are 16 subgroups! Check with Group Explorer.



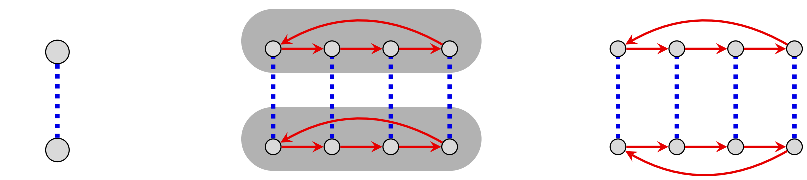
Direct products, visually

We can construct the Cayley diagram of a direct product using the following "inflation" method.

Inflation algorithm

To make a Cayley diagram of $A \times B$ from the Cayley diagrams of A and B:

1. Begin with the Cayley diagram for A.
2. Inflate each node of A, and place in each node a copy of the Cayley diagram for B.
3. Remove the (inflated) nodes of A while using the arrows of A to connect corresponding nodes from each copy of B. That is, remove the A diagram but treat its arrows as a blueprint for how to connect corresponding nodes in the copies of B.



Cyclic group $\mathbb{Z}_2$ — each node contains a copy of $\mathbb{Z}/4$ — direct product group $\mathbb{Z}/4 \times \mathbb{Z}/2$

Properties of direct products

Recall the following important definition

A subgroup $H \leq G$ is normal if $xH = Hx$ for all $x \in G$. We denote this by $H \trianglelefteq G$.

Observation

If A and B are not trivial, the direct product $A \times B$ has *at least* 4 normal subgroups:

$$\{e_A\} \times \{e_B\}, \qquad A \times \{e_B\}, \qquad \{e_A\} \times B, \qquad A \times B.$$

Proof:

Sometimes we "abuse notation": write $A \trianglelefteq A \times B$ to mean $A \times \{e_B\} \trianglelefteq A \times B$ and write $B \trianglelefteq A \times B$ to mean $\{e_A\} \times B \trianglelefteq A \times B$.

(Technically, A and B are not even subsets of $A \times B$.)

Observation

In a Cayley diagram for $A \times B$, any arrow from A commutes with any arrow from B. Algebraically, this is saying that $(a, e_b) * (e_a, b) = (a, b) = (e_a, b) * (a, e_b)$ for all elements $a \in A, b \in B$.

Multiplication tables of direct products

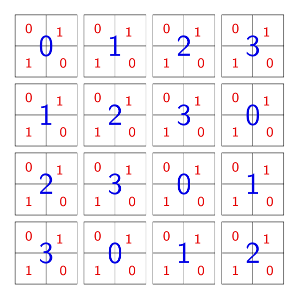
Direct products can also be visualized using multiplication tables.

For example, we construct the table for the direct product $\mathbb{Z}/4 \times \mathbb{Z}/2$:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

multipl. table for $\mathbb{Z}/4$

$\longrightarrow$



inflate each cell to contain a copy of the multiplication table of $\mathbb{Z}/2$

$\longrightarrow$

(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)
(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)	(3,1)	(3,0)
(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)	(0,0)	(0,1)
(1,1)	(1,0)	(2,1)	(2,0)	(3,1)	(3,0)	(0,1)	(0,0)
(2,0)	(2,1)	(3,0)	(3,1)	(0,0)	(0,1)	(1,0)	(1,1)
(2,1)	(2,0)	(3,1)	(3,0)	(0,1)	(0,0)	(1,1)	(1,0)
(3,0)	(3,1)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(3,1)	(3,0)	(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)

join the little tables and element names to form the table for $\mathbb{Z}/4 \times \mathbb{Z}/2$