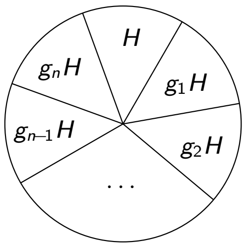
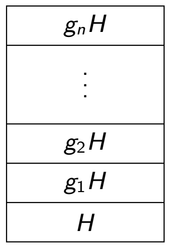
Overview

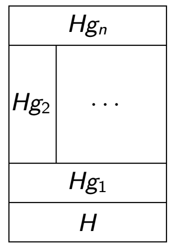
Previously, we learned that for any subgroup H of a group G,

- the left cosets of H partition G;
- the right cosets of H partition G;
- these partitions need not be the same.

Here are a few visualizations of this idea:







Subgroups whose left and right cosets agree have very special properties, and this is the topic of this lecture.

Normal subgroups

Definition

A subgroup H of G is a normal subgroup of G if $xH = Hx$ for *all* $x \in G$. We denote this as $H \lhd G$, or $H \unlhd G$.

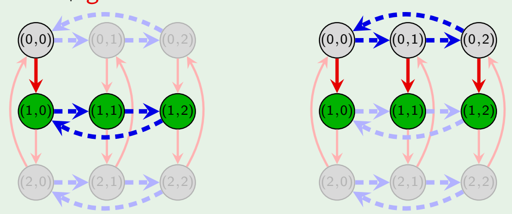
Observation from previous slides

Subgroups of abelian groups are always normal, because for any $H < G$,

$$xH = \{xh : h \in H\} = \{hx : h \in H\} = Hx\,.$$

Example

Consider the subgroup $H = \langle (0,1) \rangle = \{(0,0),(0,1),(0,2)\}$ in the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ and take $g = (1,0)$. Addition is done modulo 3, componentwise. The following depicts the equality $g + H = H + g$:



Normal subgroups of nonabelian groups

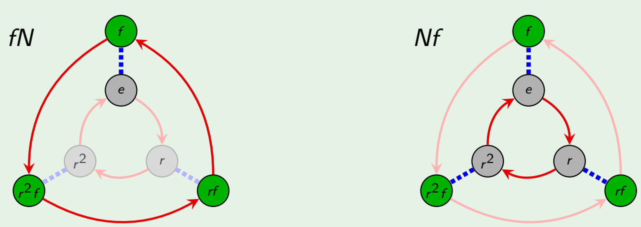
A subgroup whose left and right cosets agree is normal and has very special properties.

Since subgroups of abelian groups are always normal, we will be particularly interested in normal subgroups of non-abelian groups.

Example

Consider the subgroup $N = \{e, r, r^2\} \leq D_3$.

The cosets (left or right) of N are $N = \{e, r, r^2\}$ and $Nf = \{f, rf, r^2f\} = fN$. The following depicts this equality; the coset $fN = Nf$ are the green nodes.



Example (HW 5)

Consider the subgroup $K = \langle (1\,2)(3\,4) \rangle$ of the alternating group $G = A_4$.

1. Describe all permutations in A_4.

2. Determine whether K is normal in A_4.

Conjugate subgroups

For a fixed element $g \in G$, the set

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

is called the conjugate of H by g.

Proposition 1

For any $g \in G$, the conjugate gHg^{-1} is a subgroup of G.

Proof

1. Identity:
2. Closure:
3. Inverses: Every element $(ghg^{-1})^{-1}$ has an inverse, $gh^{-1}g^{-1}$. ✓ □

Proposition 2

$gh_1g^{-1} = gh_2g^{-1}$ if and only if $h_1 = h_2$. □

Later, you will show that H and gHg^{-1} are isomorphic subgroups.

How to check if a subgroup is normal

If $gH = Hg$, then right-multiplying both sides by g^{-1} yields $gHg^{-1} = H$.

This gives us a new way to check whether a subgroup H is normal in G.

Theorem 3

The following conditions are all equivalent to a subgroup $H \leq G$ being normal:

(i) $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");

(ii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

(iii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

Sometimes, one of these methods is *much* easier than the others!

For example, all it takes to show that H is not normal is finding *one element* $h \in H$ for which $ghg^{-1} \notin H$ for some $g \in G$.

Note: if we happen to know that G has a unique subgroup of size $|H|$, then H *must* be normal. (Why?)

Example (from HW 5)

Consider the subgroup $H = \langle(1\,2\,3)\rangle$ of $G = A_4$.

1. Find all conjugates of the subgroup H. (Hint: There are only four)

   Try $(1\,2\,4)\,H\,(1\,4\,2)$

   Try $(1\,2)(3\,4)\,H\,(1\,2)(3\,4)$

2. Is H normal in A_4?

3. Is the subgroup $J = \langle(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\rangle$ normal in A_4?

Example (from HW 5)

The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg,\ \forall g \in G\} = \{z \in G \mid gzg^{-1} = z,\ \forall g \in G\}$$

1. Prove that $Z(G)$ is a subgroup of G (similar to the proof of Prop 1)

2. Prove that $Z(G)$ is normal in G.

3. Compute the center of the following groups: C_6, D_4, D_5, Q_8, A_4, S_4.