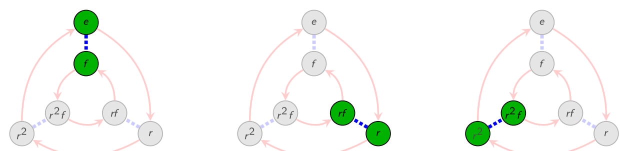
Idea of cosets

Copies of the fragment of the Cayley diagram that corresponds to a subgroup appear throughout the rest of the diagram.

Example: Below you see three copies of the fragment corresponding to the subgroup $\langle f \rangle = \{e, f\}$ in D_3.



However, only one of these copies is actually a group! Since the other two copies do *not* contain the identity, they cannot be groups.

Key concept

The elements that form these repeated copies of the fragment of a subgroup H in the Cayley diagram are called cosets of H.

Above show the three cosets of the subgroup $\{e, f\}$.

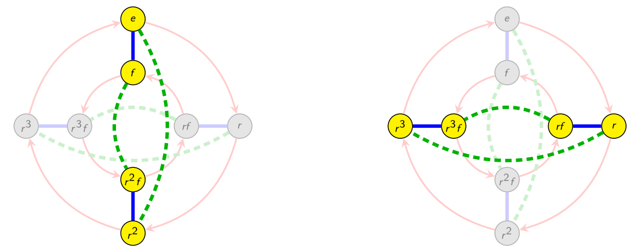
An example: D_4

Let $H = \langle f, r^2 \rangle = \{e, f, r^2, r^2f\}$, a subgroup of D_4. Find all of the cosets of H.

If we use r^2 as a generator in the Cayley diagram of D_4, then it will be easier to "see" the cosets.

Note that $D_4 = \langle r, f \rangle = \langle r, f, r^2 \rangle$. The cosets of $H = \langle f, r^2 \rangle$ are:

$$H = \langle f, r^2 \rangle = \underbrace{\{e, f, r^2, r^2f\}}_{\text{original}}, \qquad rH = r\langle f, r^2 \rangle = \underbrace{\{r, r^3, rf, r^3f\}}_{\text{copy}}.$$



Definition of cosets

Definition

If H is a subgroup of G, then a (left) coset of H is a set

$$xH = \{xh : h \in H\},$$

where $x \in G$ is some fixed element. The distinguished element (in this case, x) that we choose to use to name the coset is called the representative.

Remark

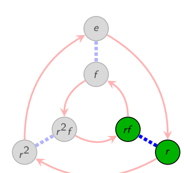
In a Cayley diagram, the (left) coset xH can be found as follows: start from node x and follow *all* paths in H.

For example, let $H = \langle f \rangle$ in D_3. The coset $\{r, rf\}$ of H is the set

$$rH = r\langle f \rangle = r\{e, f\} = \{r, rf\}.$$

Alternatively, we could have written $(rf)H$ to denote the same coset, because

$$rfH = rf\{e, f\} = \{rf, rf^2\} = \{rf, r\}.$$



More on cosets

Proposition 1

For any subgroup $H \leq G$, the union of the (left) cosets of H is the whole group G.

Proof

We only need to show that every element $x \in G$ lives in some coset of H. But, since $e \in H$ (because H is a group) and $x = xe$, we can conclude that x lives in the coset $xH = \{xh \mid h \in H\}$. □

Proposition 2 (HW)

If $y \in xH$, then $xH = yH$. □

More on cosets

Proposition 3 (HW)

All (left) cosets of a subgroup H of G have the same size as H. □

Hint: Define a bijection between $eH = H$ and another coset xH. Copy the bijection between the even permutations and odd permutations from notes 2.4, but replace (12) with x.

More on cosets

Proposition 4

For any subgroup $H \leq G$, the (left) cosets of H partition the group G.

Proof

To show that the set of (left) cosets of H form a partition of G, we need to show that (1) the union of all (left) cosets of H is equal to G, and
(2) if H is a proper subgroup, then the intersection of each pair of two distinct (left) cosets of H is empty.

Part (1) has been shown earlier in Proposition 1: every element x is the coset xH. To show part (2), suppose that $x \in G$ lies in a coset yH. Then by Proposition 2, $xH = yH$. So every element of G lives in exactly one coset. □

Subgroups also have right cosets: $Ha = \{ha \colon h \in H\}$.

For example, the three right cosets of $H = \langle f \rangle$ in D_3 are H,

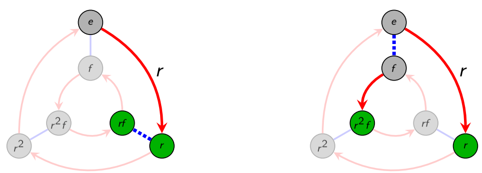
$$Hr = \langle f \rangle r = \{e, f\}r = \{r, fr = r^2f\}, \text{ and}$$
$$\langle f \rangle r^2 = \{e, f\}r^2 = \{r^2, fr^2\} = \{r^2, rf\}.$$

In this example, the left cosets for $\langle f \rangle$ are different from the right cosets.

Left vs. right cosets

The left diagram below shows the left coset $r\langle f\rangle$ in D_3: the nodes that f arrows can reach after the path to r has been followed.

The right diagram shows the right coset $\langle f\rangle r$ in D_3: the nodes that r arrows can reach from the elements in $\langle f\rangle$.



Left cosets look like copies of the subgroup, while the elements of right cosets are usually scattered (only because we adopted the convention that arrows in a Cayley diagram represent right multiplication).

Key point

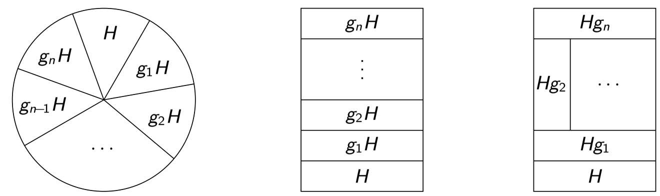
Left and right cosets are generally different.

Left vs. right cosets

For *any* subgroup $H \leq G$, we can think of G as the union of non-overlapping and equal size copies of H, namely the left cosets of H.

Though the right cosets also partition G, the corresponding partitions could be different!

Here are a few visualizations of this idea:



Definition

If $H < G$, then the index of H in G, written $[G:H]$, is the number of distinct left (or equivalently, right) cosets of H in G.

Left vs. right cosets

The left and right cosets of the subgroup $H = \langle f\rangle \leq D_3$ are *different*:

r^2H	r^2f	r^2
rH	r	rf
H	e	f

	Hr	Hr^2
	r^2f	r^2
	r	rf
H	e	f

The left and right cosets of the subgroup $N = \langle r\rangle \leq D_3$ are *the same*:

fN	f	rf	r^2f
N	e	r	r^2

Nf	f	rf	r^2f
N	e	r	r^2

Proposition 5 (HW)

If $H \leq G$ has index $[G:H] = 2$, then the left and right cosets of H are the same.

Cosets of abelian groups

Recall: in some abelian groups, we use the symbol $+$ for the binary operation.

In this case, we write the left cosets as $a+H$ (instead of aH).

For example, let $G := (\mathbb{Z}, +)$, and consider the subgroup $H := 4\mathbb{Z} = \{4k \mid k \in \mathbb{Z}\}$ of G consisting of multiples of 4. Then the left cosets of H are

$$\begin{aligned} H &= \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\} \\ 1+H &= \{\ldots, -11, -7, -3, 1, 5, 9, 13, \ldots\} \\ 2+H &= \{\ldots, -10, -6, -2, 2, 6, 10, 14, \ldots\} \\ 3+H &= \{\ldots, -9, -5, -1, 3, 7, 11, 15, \ldots\}. \end{aligned}$$

Notice that these are the same as the right cosets of H:

$$H, \qquad H+1, \qquad H+2, \qquad H+3.$$

Exercise: Why are the left and right cosets of an abelian group *always* the same?

Note that it would be confusing to write $3H$ for the coset $3+H$. In fact, $3H$ would usually be interpreted to mean the subgroup $3(4\mathbb{Z}) = 12\mathbb{Z}$.

A theorem of Joseph Lagrange

Lagrange's Theorem

Assume G is finite. If $H < G$, then $|H|$ divides $|G|$.

Proof

Suppose there are n left cosets of the subgroup H. Since they are all the same size (by Proposition 3) and they partition G (by Proposition 4), we must have

$$|G| = \underbrace{|H| + \cdots + |H|}_{n \text{ copies}} = n|H|.$$

Therefore, $|H|$ divides $|G|$. □

Corollary of Lagrange's Theorem

If G is a finite group and $H \leq G$, then

$$[G:H] = \frac{|G|}{|H|}.$$

A theorem of Joseph Lagrange

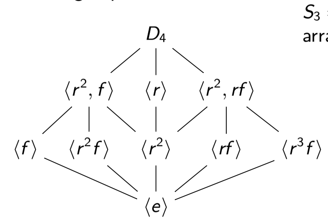
Corollary of Lagrange's Theorem

If G is a finite group and $H \leq G$, then $[G:H] = \frac{|G|}{|H|}$

This significantly narrows down the possibilities for subgroups.

Warning: The converse of Lagrange's Theorem is not generally true. That is, just because $|G|$ has a divisor d does *not* mean that there is a subgroup of order d.

The subgroup lattice of D_4:



From HW: Find all subgroups of $S_3 = \{e, (12), (23), (13), (123), (132)\}$ and arrange them in a subgroup lattice.