Math 3230 Abstract Algebra I
Sec 3.1: Subgroups

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Abstract Algebra I

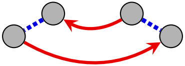
Regularity

Cayley diagrams have an important structural property called *regularity* that we've mentioned, but haven't analyzed in depth.

This is best seen with an example: Consider the group D_3. It is easy to verify that $frf = r^{-1}$.

Thus, starting at *any node* in the Cayley diagram, the path frf will *always* lead to the same node as the path r^{-1}.

That is, the following fragment permeates throughout the diagram.



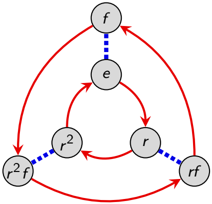
Equivalently, the path $frfr$ will always bring you back to where you started. (Because $frfr = e$).

Key observation

The algebraic relations of a group, like $frf = r^{-1}$, give Cayley diagrams a uniform symmetry – every part of the diagram is structured like every other.

Regularity

Let's look at the Cayley diagram for D_3:



Check that indeed, $frf = r^{-1}$ holds by following the corresponding paths starting at any of the six nodes.

There are other patterns that permeate this diagram, as well. Do you see any?

Here are a couple: $\quad f^2 = e, \quad r^3 = e.$

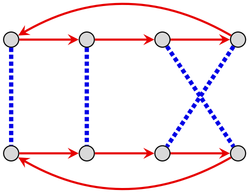
Definition

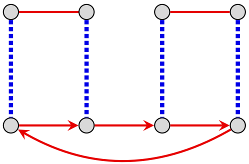
A diagram is called regular if it repeats every one of its interval patterns throughout the whole diagram, in the sense described above.

Regularity

Every Cayley diagram is regular. In particular, diagrams lacking regularity do *not* represent groups (and so they are not called Cayley diagrams).

Here are two diagrams that *cannot* be the Cayley diagram for a group because they are not regular.





Subgroups

Definition

When one group is contained in another, the smaller group is called a subgroup of the larger group. If H is a subgroup of G, we write $H < G$ or $H \leq G$.

All of the orbits that we saw in previous lectures are subgroups. Moreover, they are *cyclic* subgroups. **(Why?)**

For example, the orbit of r in D_3 is a subgroup of order 3 living inside D_3. We can write

$$\langle r \rangle = \{e, r, r^2\} < D_3.$$

In fact, since $\langle r \rangle$ is really just a copy of C_3, we may be less formal and write

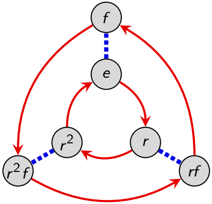
$$C_3 < D_3.$$

An example: D_3

Recall that the orbits of D_3 are

$$\langle e\rangle = \{e\}, \qquad \langle r\rangle = \langle r^2\rangle = \{e, r, r^2\}, \qquad \langle f\rangle = \{e, f\}$$
$$\langle rf\rangle = \{e, rf\}, \qquad \langle r^2 f\rangle = \{e, r^2 f\}.$$

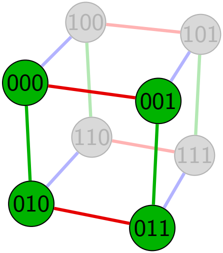
The orbits corresponding to the generators are staring at us in the Cayley diagram. The others are more hidden.



It turns out that all of the subgroups of D_3 are just (cyclic) orbits.

However, there are groups that have subgroups that are *not* cyclic.

Another example: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$



Here is the Cayley diagram for the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (the "three-light switch group").

A copy of the subgroup V_4 is highlighted.

The group V_4 requires at least two generators and hence is *not* a cyclic subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case, we can write

$$\langle 001, 010 \rangle = \{000, 001, 010, 011\} < \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Every (nontrivial) group G has *at least* two subgroups:

1. the trivial subgroup: $\{e\}$
2. the non-proper subgroup: G. (Every group is a subgroup of itself.)

Question

Which groups have *only* these two subgroups?

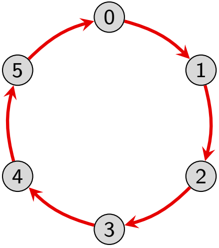
Yet one more example: $\mathbb{Z}/6$

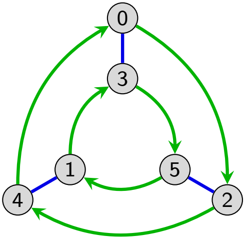
It is not difficult to see that the subgroups of $\mathbb{Z}/6 = \{0, 1, 2, 3, 4, 5\}$ are

$$\langle 0 \rangle = \{0\}, \qquad \langle 2 \rangle = \langle 4 \rangle = \{0, 2, 4\}, \qquad \langle 3 \rangle = \{0, 3\}, \qquad \langle 1 \rangle = \langle 5 \rangle = \mathbb{Z}_6.$$

Depending on our choice of generators and layout of the Cayley diagram, not all of these subgroups may be "visually obvious."

Here are two Cayley diagrams for $\mathbb{Z}/6$, one generated by $\langle 1 \rangle$ and the other by $\langle 2, 3 \rangle$:





Another example: D_4

The dihedral group D_4 has 10 subgroups (though some are isomorphic to each other):

$$\{e\}, \underbrace{\langle r^2\rangle, \langle f\rangle, \langle rf\rangle, \langle r^2 f\rangle, \langle r^3 f\rangle}_{\text{order 2}}, \underbrace{\langle r\rangle, \langle r^2, f\rangle, \langle r^2, rf\rangle}_{\text{order 4}}, D_4.$$

We can arrange the subgroups in a diagram called a subgroup lattice that shows which subgroups contain other subgroups.

The subgroup lattice of D_4:

$$\begin{tikzcd}
& & D_4 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
& \langle r^2, f\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \langle r\rangle \arrow[d, dash] & \langle r^2, rf\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle f\rangle \arrow[drr, dash] & \langle r^2 f\rangle \arrow[dr, dash] & \langle r^2\rangle \arrow[d, dash] & \langle rf\rangle \arrow[dl, dash] & \langle r^3 f\rangle \arrow[dll, dash] \\
& & \langle e\rangle & &
\end{tikzcd}$$

Exercise (from HW 4): Find all subgroups of $S_3 = \{e, (12), (23), (13), (123), (132)\}$ and arrange them in a subgroup lattice.

A (terrible) way to find all subgroups

Here is a brute-force method for finding all subgroups of a given group G of order n.

Though this algorithm is horribly inefficient, it makes a good thought exercise.

0. we always have $\{e\}$ and G as subgroups
1. find all subgroups generated by a single element ("cyclic subgroups")
2. find all subgroups generated by 2 elements

$$\vdots$$

n-1. find all subgroups generated by $n-1$ elements

Along the way, we will certainly duplicate subgroups; one reason why this is so inefficient and impractible.

This algorithm works because every group (and subgroup) has a set of generators.

Soon, we will see how a result known as Lagrange's theorem greatly narrows down the possibilities for subgroups.