Math 3230 Abstract Algebra I
Sec 2.4: Cayley's Theorem

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http://egunawan.github.io/algebra

Abstract Algebra I

Overview

We have introduced these 5 families of groups:

1. cyclic groups
2. abelian groups
3. dihedral groups
4. symmetric groups
5. alternating groups

In this lecture, we will introduce Cayleys theorem, which tells us that every finite group is isomorphic to a collection of permutations (i.e., a subgroup of a symmetric group).

Cayley's theorem

Any set of permutations that forms a group is called a permutation group.

Cayley's theorem says that permutations can be used to construct any finite group.

In other words, every group has the same structure as (we say "*is isomorphic to*") some permutation group.

Warning! We are *not* saying that every group is isomorphic to some symmetric group, S_n. Rather, every group is isomorphic to a subgroup of some symmetric group S_n – i.e., a subset of S_n that is *also* a group in its own right.

Question

Given a group, how do we associate it with a set of permutations?

Cayley's theorem (an algorithm for constructing permutations)

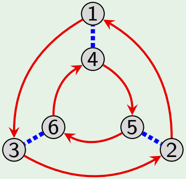
Given a Cayley diagram with n nodes:

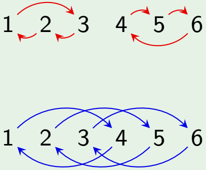
1. number the nodes 1 through n,
2. interpret each arrow type in the Cayley diagram as a permutation.

The resulting permutations are the generators of the corresponding permutation group.

Example

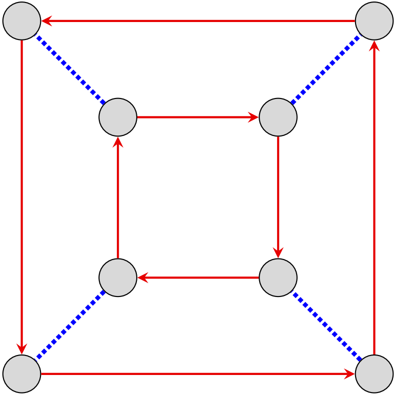
Follow the algorithm with $D_3 = \langle r, f \rangle$.





We see that D_3 is isomorphic to the subgroup $\langle (132)(456), (14)(25)(36) \rangle$ of S_6.
Exercise: Compute the other elements (as permutations), for example,
$(132)(456)\ (14)(25)(36) =$

Exercise A (from HW4)



- Label the vertices of the Cayley diagram from the set $\{1, \ldots, 8\}$ and use this to construct a permutation group isomorphic to D_4, and sitting inside S_8.

Cayley's theorem (an algorithm to construct permutations)

Here is an algorithm given a multiplication table with n elements:

1. replace the table headings with 1 through n,
2. make the appropriate replacements throughout the rest of the table,
3. interpret each row as a permutation.

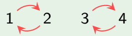
This gives a bijection between the original group elements (not just the generators) and permutations.

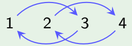
Example

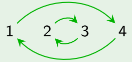
Try this with the multiplication table for $V_4 = \langle v, h \rangle$.

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Row 1: 1 2 3 4

Row 2:


Row 3:


Row 4:


We see that V_4 is isomorphic to the subgroup $\langle (12)(34), (13)(24) \rangle$ of S_4. The other two permutations in this subgroup are (

Exercise B (from HW4)

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

1. Label the entries of the multiplication table from the set $\{1, \ldots, 8\}$ and use this to construct a permutation group isomorphic to D_4, and sitting inside S_8.
2. Are the two groups you got the same as the previous exercise? (The answer will depend on your choice of labeling.) If "yes", then repeat the previous exercise with a different labeling to yield a different group. If "no", then repeat the previous exercise with a different labeling to yield the group you got.

Cayley's theorem

Intuitively, two groups are isomorphic if they have the same structure.

Two groups are *isomorphic* if we can construct Cayley diagrams for each that look identical.

Cayley's Theorem

Every finite group is isomorphic to a collection of permutations.

Our algorithms exhibit a 1-1 correspondence between group elements and permutations.

However, we have *not* shown that the corresponding permutations form a group, or that the resulting permutation group has the same structure as the original.

We need to show that the permutation from the i^{th} row followed by the permutation from the j^{th} column, results in the permutation that corresponding to the cell in the i^{th} row and j^{th} column of the original table.