Math 3230 Abstract Algebra I
Sec 2.3: Symmetric and alternating groups

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Abstract Algebra I
Overview

In this section, we will introduce 5 families of groups:

1. cyclic groups
2. abelian groups
3. dihedral groups
4. symmetric groups
5. alternating groups

This lecture is focused on the last two families: symmetric groups and alternating groups.

A symmetric group is the collection of all \(n!\) permutations of \(n\) objects.

We will study permutations, and how to write them conciselv in cycle notation.

Cayley’s theorem tells us that every finite group is isomorphic to a collection of permutations (i.e., a **subgroup** of a symmetric group).
Definition

A permutation is an action that rearranges a collection of objects.

For convenience, we will usually refer to permutations of positive integers (just like we did when we numbered our rectangle, etc.).

There are many ways to represent permutations, but we will start with the notation illustrated by the following example.

Example

Here are some permutations of 4 objects.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ
\end{array}
\]
Combining permutations

In order for the set of permutations of \( n \) objects to form a group (what we want!), we need to understand how to combine permutations.

Example:

What should

\[
\begin{array}{c}
1 \\
\circlearrowleft
\end{array}
\quad \begin{array}{c}
2 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
3 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
4 \\
\circlearrowright
\end{array}
\]

followed by

\[
\begin{array}{c}
1 \\
\circlearrowleft
\end{array}
\quad \begin{array}{c}
2 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
3 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
4 \\
\circlearrowright
\end{array}
\]

be equal to?

The first permutation rearranges the 4 objects, and then we shuffle the result according to the second permutation:

\[
\begin{array}{c}
1 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
2 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
3 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
4 \\
\circlearrowright
\end{array}
\] \quad \ast \quad \begin{array}{c}
1 \\
\circlearrowleft
\end{array}
\quad \begin{array}{c}
2 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
3 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
4 \\
\circlearrowright
\end{array}
\]

\[
\begin{array}{c}
1 \\
\circlearrowleft
\end{array}
\quad \begin{array}{c}
2 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
3 \\
\circlearrowright
\end{array}
\quad \begin{array}{c}
4 \\
\circlearrowright
\end{array}
\]

Remember to read from left to right!
Groups of permutations

Fact

There are \( n! = n(n-1) \cdots 3 \cdot 2 \cdot 1 \) permutations of \( n \) items.

For example, there are \( 4! = 24 \) “permutation pictures” on 4 objects.

The collection of permutations of \( n \) items forms a group!

To verify this, we just have to check that the appropriate rules of one of our definitions of a group hold.

How do we find the inverse of a permutation? Just reverse all of the arrows in the permutation picture. For example, the inverse of

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

is simply

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]
The symmetric group

**Definition**

The group of all permutations of \( n \) items is called the symmetric group (on \( n \) objects) and is denoted by \( S_n \).

We’ve already seen the group \( S_3 \), which happens to be the same as the dihedral group \( D_3 \), but this is the only time the symmetric groups and dihedral groups coincide. (*Why?)

Although the set of *all* permutations of \( n \) items forms a group, creating a group does not require taking all permutations.

If we choose carefully, we can form groups by taking a subset of the permutations.

For example, the cyclic group \( C_n \) and the dihedral group \( D_n \) can both be thought of groups of certain permutations of \( \{1, \ldots, n\} \). (*Why? Do you see which permutations they represent?)
Cycle notation for $S_n$

We can concisely describe the permutation

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow 4
\]

as \((1 \ 2 \ 3 \ 4)\).

This is called cycle notation.

Observation 1

Every permutation can be decomposed into a product of disjoint cycles.

For example, in $S_{10}$, we can write

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10
\]

as \((1 \ 4 \ 6 \ 5) \ (2 \ 3) \ (8 \ 10 \ 9)\).

Observation 2

Disjoint cycles commute.

For example:

\[
(1 \ 4 \ 6 \ 5) \ (2 \ 3) \ (8 \ 10 \ 9) = (2 \ 3) \ (8 \ 10 \ 9) \ (1 \ 4 \ 6 \ 5).
\]
Cycle notation for $S_n$

**Example**

Consider the following permutations in $S_4$:

- $\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$ is $(1\ 2)\ (3\ 4)$
- $\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$ is $(2\ 3)$
- $\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$ is $(1\ 3)\ (2\ 4)$
- $\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$ is $(1\ 3\ 2)$

**Remark**

It doesn’t matter “where we start” when writing the cycle. In the last example above,

$$(1\ 3\ 2) = (3\ 2\ 1) = (2\ 1\ 3) = (1\ 2)\ (2\ 3) = (1\ 2)\ (2\ 3)\ (2\ 3)\ (2\ 3).$$
Composing permutations in cycle notation

Recall how we combined permutations:

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\end{array} \quad \ast \quad \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\end{array} = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\end{array} \]

In cycle notation, this is

\[(1 \ 2 \ 3 \ 4) \ast (1 \ 3) \ (2 \ 4) = (1 \ 4 \ 3 \ 2).\]

We read left-to-right. (Caveat: some books use the right-to-left convention as in function composition.)

Do you see how to combine permutations in cycle notation? In the example above, we start with 1 and then read off:

- “1 goes to 2, then 2 goes to 4”; Write: (1 4
- “4 goes to 1, then 1 goes to 3”; Write: (1 4 3
- “3 goes to 4, then 4 goes to 2”; Write: (1 4 3 2
- “2 goes to 3, then 3 goes to 1”; Write: (1 4 3 2

In this case, we’ve used up each number in \{1, \ldots, n\}. If we hadn’t, we’d take the the smallest unused number and continue the process with a new (disjoint) cycle.
Transpositions

A transposition is a permutation that swaps two objects and fixes the rest, e.g.:

\[
1 \ 2 \ \cdots \ i-1 \ i \ i+1 \ \cdots \ j-1 \ j \ j+1 \ \cdots \ n-1 \ n
\]

In cycle notation, a transposition is just a 2-cycle, e.g., \((i \ j)\).

Theorem

The group \(S_n\) is generated by transpositions.

Intuitively, this means that every permutation can be constructed by successively exchanging pairs of objects.

In other words, if \(n\) people are standing in a row, and we want to rearrange them in some other order, we can always do this by successively having pairs of people swap places.

In fact, we only need adjacent transpositions to generate \(S_n\):

\[
S_n = \langle (1 \ 2), (2 \ 3), \ldots, (n-1 \ n) \rangle.
\]
Transpositions and the alternating groups

Remark

Even though every permutation in $S_n$ can be written as a product of transpositions, there may be many ways to do this.

For example:

$$(1\ 3\ 2) = (1\ 2)\ (2\ 3) = (1\ 2)\ (2\ 3)\ (2\ 3) = (1\ 2)\ (2\ 3)\ (1\ 2)\ (1\ 2).$$

Theorem

The parity of the number of transpositions of a fixed permutation is unique.

That is, a fixed permutation can either be written with an even number of transpositions, or an odd number of transpositions, but not both!

We thus have a notion of even permutations and odd permutations.

Theorem

Exactly half of the permutations in $S_n$ are even, and they form a group called the alternating group, denoted $A_n$. 
At this point, it helps to “get your hands dirty” and try a few examples. Here are some good exercises.

1. Write the following products of permutations into a product of disjoint cycles:
   - \((1 \, 2 \, 3) \, (1 \, 2 \, 3 \, 4)\) in \(S_4\)
   - \((1 \, 6) \, (1 \, 2 \, 4 \, 5) \, (1 \, 6 \, 4 \, 2 \, 5 \, 3)\) in \(S_6\).

2. Do the following for each element in \(S_3\):
   - Draw its “permutation picture.”
   - Write it as a product of disjoint transpositions (that is, using only \((1 \, 2)\), \((2 \, 3)\), and \((1 \, 3)\)).
   - Write it as a product of disjoint adjacent transpositions (that is, using only \((1 \, 2)\) and \((2 \, 3)\)).
   - Determine whether it is even or odd.

3. Now, write down the alternating group \(A_3\). This is the group consisting of only the even permutations. What familiar group is this isomorphic to?
Alternating groups

How can we verify that $A_n$ a group?

The only major concern is it must be closed under combining permutations (all other necessary properties are inherited from $S_n$).

Do you see why combining two even permutations yields an even permutation?

**Interesting fact**

For $n \leq 5$, the group $A_n$ consists precisely of the set of “squares” in $S_n$. By “square,” we mean an element that can be written as an element of $S_n$ times itself.

For example, the permutation $\begin{array}{c} 1 \\ \circlearrowleft \\
2 \\ \circlearrowleft \\
3 \\ \circlearrowleft \end{array}$ is a square in $S_3$, because:

\[
\begin{array}{c}
1 \\ \circlearrowleft \\
2 \\ \circlearrowleft \\
3 \\ \circlearrowleft \end{array} \times \begin{array}{c}
1 \\ \circlearrowleft \\
2 \\ \circlearrowleft \\
3 \\ \circlearrowleft \end{array} = \begin{array}{c}
1 \\ \circlearrowleft \\
2 \\ \circlearrowleft \\
3 \\ \circlearrowleft \end{array}
\]

In cycle notation, this is $(1 \ 3 \ 2) = (1 \ 2 \ 3)(1 \ 2 \ 3)$.

Note that $A_n$ has order $\frac{n!}{2}$. 
Platonic solids

The symmetric groups and alternating groups arise throughout group theory. In particular, the groups of symmetries of the 5 Platonic solids are symmetric and alternating groups.

A 3-dimensional Platonic solid is a polytope with regular polygons as faces where all angles are equal and all sides are equal. There are only five 3-dimensional platonic solides:

<table>
<thead>
<tr>
<th>shape</th>
<th>group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>$A_4$</td>
</tr>
<tr>
<td>Cube</td>
<td>$S_4$</td>
</tr>
<tr>
<td>Octahedron</td>
<td>$S_4$</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>$A_5$</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>$A_5$</td>
</tr>
</tbody>
</table>

The groups of symmetries of the Platonic solids are as follows:
Platonic solids

The Cayley diagrams for these 3 groups can be arranged in some very interesting configurations.

In particular, the Cayley diagram for Platonic solid ‘X’ can be arranged on a truncated ‘X’, where truncated refers to cutting off some corners.

For example, here are two representations for Cayley diagrams of $A_5$. At left is a truncated icosahedron and at right is a truncated dodecahedron.