Math 3230 Abstract Algebra I
Sec 2.3: Symmetric and alternating groups

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Abstract Algebra I

Overview

In this section, we will introduce 5 families of groups:

1. cyclic groups
2. abelian groups
3. dihedral groups
4. symmetric groups
5. alternating groups

This lecture is focused on the last two families: symmetric groups and alternating groups.

A symmetric group is the collection of all n! permutations of n objects.

We will study permutations, and how to write them concisely in cycle notation.

Cayley's theorem tells us that every finite group is isomorphic to a collection of permutations (i.e., a subgroup of a symmetric group).

Symmetric groups

Definition

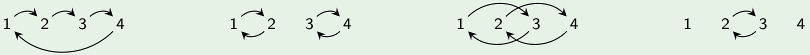
A permutation is an action that rearranges a collection of objects.

For convenience, we will usually refer to permutations of positive integers (just like we did when we numbered our rectangle, etc.).

There are many ways to represent permutations, but we will start with the notation illustrated by the following example.

Example

Here are some permutations of 4 objects.

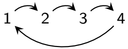


Combining permutations

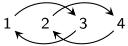
In order for the set of permutations of n objects to form a group (what we want!), we need to understand how to combine permutations.

Example:

What should

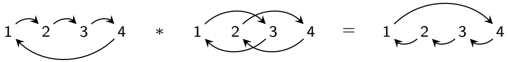


followed by



be equal to?

The first permutation rearranges the 4 objects, and then we shuffle the result according to the second permutation:



Remember to read from left to right!

Groups of permutations

Fact

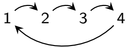
There are $n! = n(n-1)\cdots 3\cdot 2\cdot 1$ permutations of n items.

For example, there are $4! = 24$ "permutation pictures" on 4 objects.

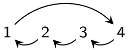
The collection of permutations of n items forms a group!

To verify this, we just have to check that the appropriate rules of *one* of our definitions of a group hold.

How do we find the inverse of a permutation? Just reverse all of the arrows in the permutation picture. For example, the inverse of



is simply



The symmetric group

Definition

The group of all permutations of n items is called the symmetric group (on n objects) and is denoted by S_n.

We've already seen the group S_3, which happens to be the same as the dihedral group D_3, but this is the only time the symmetric groups and dihedral groups coincide. (*Why?*)

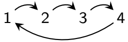
Although the set of *all* permutations of n items forms a group, creating a group does not require taking all permutations.

If we choose carefully, we can form groups by taking a subset of the permutations.

For example, the cyclic group C_n and the dihedral group D_n can both be thought of groups of certain permutations of $\{1, \ldots, n\}$. (*Why?* Do you see *which* permutations they represent?)

Cycle notation for S_n

We can concisely describe the permutation



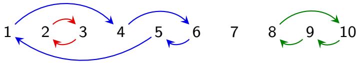
as $(1\ 2\ 3\ 4)$.

This is called cycle notation.

Observation 1

Every permutation can be decomposed into a product of disjoint cycles.

For example, in S_{10}, we can write



as $(1\ 4\ 6\ 5)\,(2\ 3)\,(8\ 10\ 9)$.

Observation 2

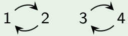
Disjoint cycles commute.

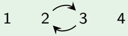
For example: $(1\ 4\ 6\ 5)\,(2\ 3)\,(8\ 10\ 9) = (2\ 3)\,(8\ 10\ 9)\,(1\ 4\ 6\ 5)$.

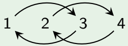
Cycle notation for S_n

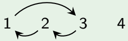
Example

Consider the following permutations in S_4:

- 

is $(1\ 2)\,(3\ 4)$
- 

is $(2\ 3)$
- 

is $(1\ 3)\ (2\ 4)$
- 

is $(1\ 3\ 2)$

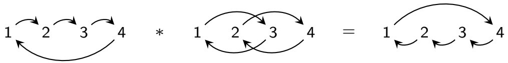
Remark

It doesn't matter "where we start" when writing the cycle. In the last example above,

$$(1\ 3\ 2) = (3\ 2\ 1) = (2\ 1\ 3) = (1\ 2)\,(2\ 3) = (1\ 2)\,(2\ 3)\,(2\ 3)\,(2\ 3)\,.$$

Composing permutations in cycle notation

Recall how we combined permutations:



In cycle notation, this is

$$(1\;2\;3\;4) * (1\;3)\,(2\;4) = (1\;4\;3\;2)\,.$$

We read left-to-right. (*Caveat*: some books use the right-to-left convention as in function composition.)

Do you see how to combine permutations in cycle notation? In the example above, we start with 1 and then read off:

- "1 goes to 2, then 2 goes to 4"; Write: (1 4
- "4 goes to 1, then 1 goes to 3"; Write: (1 4 3
- "3 goes to 4, then 4 goes to 2"; Write: (1 4 3 2
- "2 goes to 3, then 3 goes to 1"; Write: (1 4 3 2)

In this case, we've used up each number in $\{1, \ldots, n\}$. If we hadn't, we'd take the the smallest unused number and continue the process with a new (disjoint) cycle.

Transpositions

A transposition is a permutation that swaps two objects and fixes the rest, e.g.:

$$\begin{tikzcd} 1 & 2 & \cdots & i-1 & i \arrow[rrrr, bend left] & i+1 & \cdots & j-1 & j \arrow[llll, bend left] & j+1 & \cdots & n-1 & n \end{tikzcd}$$

In cycle notation, a transposition is just a 2-cycle, e.g., $(i\ \ j)$.

Theorem

The group S_n is generated by transpositions.

Intuitively, this means that every permutation can be constructed by successively exchanging pairs of objects.

In other words, if n people are standing in a row, and we want to rearrange them in some other order, we can always do this by successively having pairs of people swap places.

In fact, we only need adjacent transpositions to generate S_n:

$$S_n = \big\langle\, (1\ 2)\,,\ (2\ 3)\,,\ \ldots,\ (n-1\ \ n)\,\big\rangle\,.$$

Transpositions and the alternating groups

Remark

Even though every permutation in S_n can be written as a product of transpositions, there may be many ways to do this.

For example:

$$(1\,3\,2) = (1\,2)\,(2\,3) = (1\,2)\,(2\,3)\,(2\,3)\,(2\,3) = (1\,2)\,(2\,3)\,(1\,2)\,(1\,2).$$

Theorem

The parity of the number of transpositions of a fixed permutation is unique.

That is, a fixed permutation can either be written with an even number of transpositions, or an odd number of transpositions, but *not both*!

We thus have a notion of even permutations and odd permutations.

Theorem

Exactly half of the permutations in S_n are even, and they form a group called the alternating group, denoted A_n.

Practice

At this point, it helps to "get your hands dirty" and try a few examples. Here are some good exercises.

1. Write the following products of permutations into a product of *disjoint cycles*:
   - $(1\ 2\ 3)(1\ 2\ 3\ 4)$ in S_4
   - $(1\ 6)(1\ 2\ 4\ 5)(1\ 6\ 4\ 2\ 5\ 3)$ in S_6.

Let $G = S_3$, the symmetric group on three objects. This group has six elements.

2. Do the following for each element in S_3:
   - Draw its "permutation picture."
   - Write it as a product of disjoint transpositions (that is, using only $(1\ 2)$, $(2\ 3)$, and $(1\ 3)$).
   - Write it as a product of disjoint *adjacent* transpositions (that is, using only $(1\ 2)$ and $(2\ 3)$).
   - Determine whether it is even or odd.
3. Now, write down the alternating group A_3. This is the group consisting of only the *even* permutations. What familiar group is this isomorphic to?

Alternating groups

How can we verify that A_n a group?

The only major concern is it must be closed under combining permutations (all other necessary properties are inherited from S_n).

Do you see why combining two even permutations yields an even permutation?

Interesting fact

For $n \leq 5$, the group A_n consists precisely of the set of "squares" in S_n. By "square," we mean an element that can be written as an element of S_n times itself.

For example, the permutation  is a square in S_3, because:

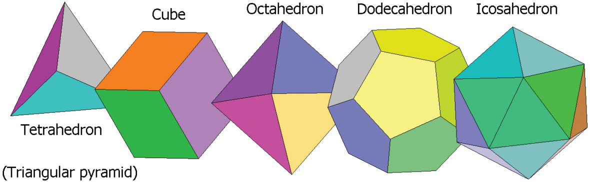
In cycle notation, this is $(1\ 3\ 2) = (1\ 2\ 3)\,(1\ 2\ 3)$.

Note that A_n has order $\dfrac{n!}{2}$.

Platonic solids

The symmetric groups and alternating groups arise throughout group theory. In particular, the groups of symmetries of the 5 Platonic solids are symmetric and alternating groups.

A 3-dimensional *Platonic solid* is a polytope with regular polygons as faces where all angles are equal and all sides are equal. There are only *five* 3-dimensional platonic solides:



The groups of symmetries of the Platonic solids are as follows:

shape	group
Tetrahedron	A_4
Cube	S_4
Octahedron	S_4
Icosahedron	A_5
Dodecahedron	A_5

Platonic solids

The Cayley diagrams for these 3 groups can be arranged in some very interesting configurations.

In particular, the Cayley diagram for Platonic solid 'X' can be arranged on a truncated 'X', where truncated refers to cutting off some corners.

For example, here are two representations for Cayley diagrams of A_5. At left is a truncated icosahedron and at right is a truncated dodecahedron.