Math 3230 Abstract Algebra I
Section 1.4 Multiplication Tables

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http://egunawan.github.io/algebra

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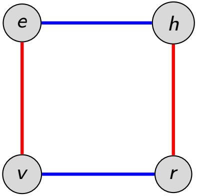
Multiplication tables

Since we can use a Cayley diagram with nodes labeled by actions as a "group calculator," we can create a (group) multiplication table, that shows how every pair of group actions combine.

Order of multiplication can matter, so we pick the convention that the entry in row g and column h is the element gh.

Example: a multiplication table for V_4.



	e	v	h	r
e	e	v	h	r
v	v	e	r	h
h	h	r	e	v
r	r	h	v	e

- Group Explorer software: `https://nathancarter.github.io/group-explorer/GroupExplorer.html`

Some remarks on the structure of multiplication tables

Comments

- The 1st column and 1st row repeat themselves. (Why?) Sometimes these will be omitted (*Group Explorer* does this).
- Multiplication tables can visually reveal patterns that may be difficult to see otherwise. Color the boxes to make these patterns more obvious.
- A group is abelian iff its multiplication table is symmetric about the "main diagonal."
- In each row and each column, each group action occurs exactly once. (This will always happen. Why? See the next slide)

A theorem and proof

Theorem

An element cannot appear twice in the same row of a multiplication table.

Proof

Suppose that in row a, the element g appears in columns b and c. Algebraically, this means

$$ab = g = ac.$$

Multiplying everything on the left by a^{-1} yields

$$a^{-1}ab = a^{-1}g = a^{-1}ac.$$

This implies that $b = c$.
Thus, the element g cannot appear twice in the same row.

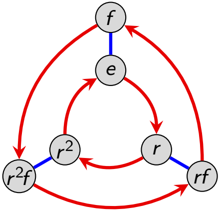
$\square$

The proof that two elements cannot appear twice in the same column is similar, and will be left as a homework exercise.

Another example: D_3

Let's fill out a multiplication table for the group D_3; here are several different presentations:

$$\begin{aligned} D_3 &= \langle r, f \mid r^3 = e,\ f^2 = e,\ rf = fr^2 \rangle \\ &= \langle r, f \mid r^3 = e,\ f^2 = e,\ rfr = f \rangle \,. \end{aligned}$$



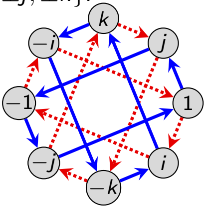
	e	r	r^2	f	rf	r^2f
e	e	r	r^2	f	rf	r^2f
r	r	r^2	e	rf	r^2f	f
r^2	r^2	e	r	r^2f	f	rf
f	f	r^2f	rf	e	r^2	r
rf	rf	f	r^2f	r	e	r^2
r^2f	r^2f	rf	f	r^2	r	e

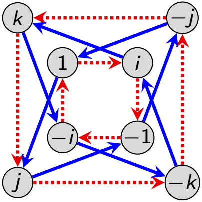
Observations? What patterns do you see?

Just for fun, what group do you get if you remove the "$r^3 = e$" relation from the presentations above? (*Hint*: We've seen it recently!)

Another example: the Quaternion group Q_8

Below is the *same* Cayley diagram which describes the Quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.





The elements j and k individually "act like" $i = \sqrt{-1}$, because $i^2 = j^2 = k^2 = -1$.

Multiplication of $\{\pm i, \pm j, \pm k\}$ works like the cross product of unit vectors in $\mathbb{R}^3$:

$$ij = k, \quad jk = i, \quad ki = j, \qquad\qquad ji = -k, \quad kj = -i, \quad ik = -j\,.$$

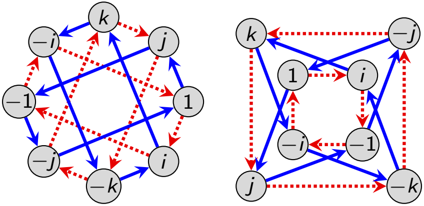
Here are two possible group presentations for Q_8:

$$\begin{aligned} Q_8 &= \langle x, y, z \mid x^2 = y^2 = k^2 = xyz = -1 \rangle \\ &= \langle x, y \mid x^4 = y^4 = 1,\ xyx = y \rangle\,. \end{aligned}$$

Question

One of these presentations corresponds to the above Cayley diagram. Which one?

Multiplication Table for the Quaternion group Q_8



	1	i	j	k	-1	$-i$	$-j$	$-k$
1								
i			k	$-j$				
j			-1					
k								
-1								
$-i$								
$-j$								
$-k$								

$$i^2 = j^2 = k^2 = -1 \qquad ij = k, \quad jk = i, \quad ki = j, \qquad ji = -k, \quad kj = -i, \quad ik = -j\,.$$

Verify with Group Explorer

https://nathancarter.github.io/group-explorer/GroupExplorer.html