Math 3230 Abstract Algebra I
Section 1.3 Inverses and group presentations

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Labeled Cayley diagrams

When we have been drawing Cayley diagrams, we have been doing one of two things with the nodes:

1. Labeling the nodes with configurations of a thing we are acting on.
2. Leaving the nodes unlabeled (this is the "abstract Cayley diagram").

Recall: every path in the Cayley diagram represents an action of the group. Today, we focus on doing the following with the nodes:

3. Label the nodes with actions (this is called a "diagram of actions").

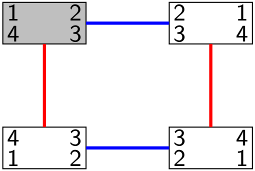
Node Labeling Algorithm

Distinguish one node with the identity action, e. Label each remaining node in with a path that leads there from node e. (If there is more than one path, pick any one; shorter is better.)

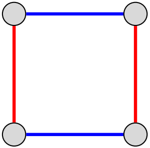
Labeled Cayley diagram example: The Klein 4-group

Recall the "rectangle puzzle." We may choose (among 3 possible minimal generating sets) the horizontal flip (h) and vertical flip (v) as generators.

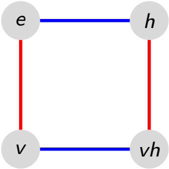
nodes labeled by configurations

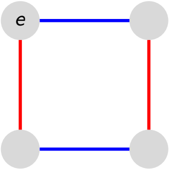


nodes unlabeled
(abstract Cayley diagram)



From the abstract Cayley diagram for V_4, create a "diagram of actions" using the upper-left node as the "identity" node:





Inverses

If g is a generator in a group G, then following the "g-arrow" backwards is an action that we call its inverse, and denoted by g^{-1}.

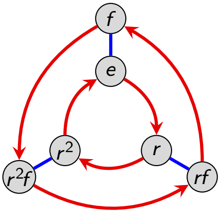
More generally, if g (not necessarily a generator) is represented by a path in a Cayley diagram, then g^{-1} is the action achieved by tracing out this path in reverse.

Note that by construction,

$$gg^{-1} = g^{-1}g = e\,,$$

where e is the identity (or "do nothing") action, often denoted by e, 1, or 0.

Exercise: Use the following Cayley diagram to compute the inverses of a few actions:



$r^{-1} = \_\_\_\_\_$ because $r\_\_\_\_\_ = e = \_\_\_\_\_r$

$f^{-1} = \_\_\_\_\_$ because $f\_\_\_\_\_ = e = \_\_\_\_\_f$

$(rf)^{-1} = \_\_\_\_\_$ because $(rf)\_\_\_\_\_ = e = \_\_\_\_\_(rf)$

$(r^2f)^{-1} = \_\_\_\_\_$ because $(r^2f)\_\_\_\_\_ = e = \_\_\_\_\_(r^2f)$.

Socks-Shoes property

Theorem

For all group actions a and b, we have $(ab)^{-1} = b^{-1}a^{-1}$.

Proof.

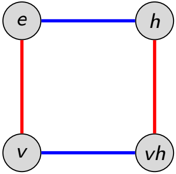
$\square$

A "group calculator"

One neat thing about Cayley diagrams with nodes labeled by actions is that they act as a "group calculator".

For example, if we want to know what a particular sequence is equal to, we can just chase the sequence through the Cayley graph, starting at e.

Let's try one. In V_4, what is the action $hhhvhvvhv$ equal to?



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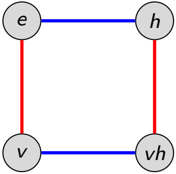


We see that $hhhvhvvhv = h$. A more condensed way to write this is

$$hhhvhvvhv = h^3vhv^2hv = h.$$

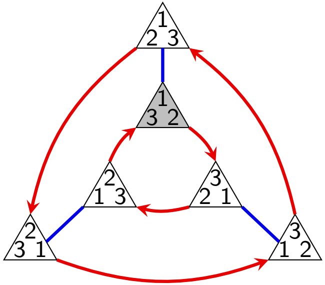
A concise way to describe V_4 is by the following group presentation:

$$V_4 = \langle v, h \mid v^2 = e, h^2 = e, vh = hv \rangle.$$

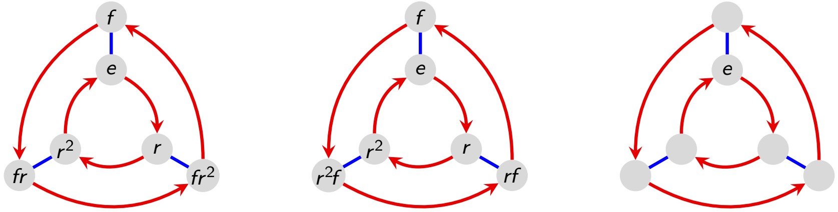
Another familiar example: D_3

Recall the "triangle puzzle" group $G = \langle r, f \rangle$, which can be generated by a clockwise 120° rotation r, and a horizontal flip f.

Here, we choose to label node with the shaded triangle with the identity e.



Here are some different ways (of many!) that we can label the nodes with actions:



The following is one (of many!) presentations for this group:

$$D_3 = \langle r, f \mid r^3 = e,\ f^2 = e,\ r^2 f = fr \rangle .$$
$$= \langle r, f \mid r^3 = e,\ f^2 = e,\ rfr = f \rangle .$$

Another Cayley diagram for D_3

Recall from homework another minimal set of generators of D_3, the flips f and g.

Abstract Cayley diagram using f, g:

Some possible diagrams of actions:

Some possible group presentations using this abstract Cayley diagram:

Group presentations

Previously, we wrote $G = \langle h, v \rangle$ to say that "G is generated h and v."

This tells us is that h and v generate G, but not how they generate G.

To be more precise, use a group presentation:

$$G = \Big\langle \text{generators} \;\Big|\; \text{relations} \Big\rangle$$

Think of the vertical bar as "subject to" or "such as".

For example, the following is a presentation for V_4:

$$V_4 = \langle a, b \mid a^2 = e,\ b^2 = e,\ ab = ba \rangle .$$

Caveat!

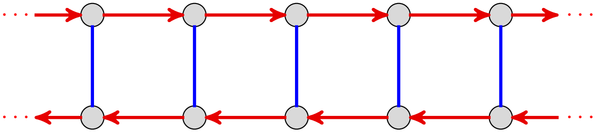
Just because there are actions in a group that "satisfy" the relations above does *not* mean that it is V_4.

E.g., the trivial group $G = \{e\}$ satisfies the above presentation; take $a = e$, $b = e$.

Loosely speaking, the above presentation tells us that V_4 is the "largest group" that satisfies these relations. (More on this later when we study quotients.)

Group presentations (Example from frieze groups)

Recall the following Cayley diagram (from frieze groups):

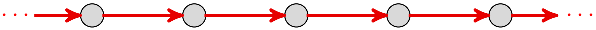


One possible presentation of this group is

$$G = \langle T, f \mid f^2 = e, T\,f\,T = f \rangle .$$

Group presentations (Another example from frieze groups)

Here is the Cayley diagram of another frieze group:



It has presentation

$$G = \langle a \mid \quad \rangle \, .$$

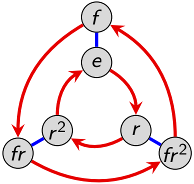
That is, "one generator subject to *no relations*."

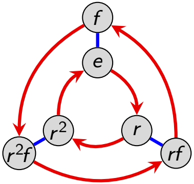
The problem (called the word problem) of determining what a mystery group is from a presentation is actually computationally unsolvable! In fact, it is equivalent to the famous "halting problem" in computer science.

Back to D_3

Two different possible presentations:

$$D_3 = \langle r, f \mid r^3 = e,\ f^2 = e,\ rf = fr^2 \rangle$$
$$= \langle r, f \mid r^3 = e,\ f^2 = e,\ rfr = f \rangle\,.$$



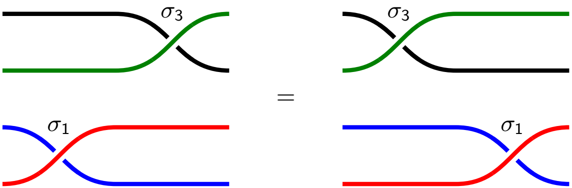


Exercise:

What group do you get if you remove the "$r^3 = e$" relation from the presentations above? (*Hint*: We've seen it recently!)

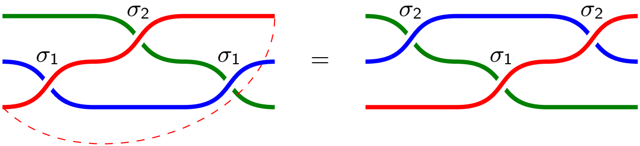
Group presentations (Example from braid groups)

There are two fundamental relations in braid groups:



$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

(if $|i - j| \geq 2$)



$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

We can describe the braid group B_4 by the following presentation:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1, \;\; \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \;\; \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle.$$