Math 3230 Abstract Algebra I
Section 1.2 Groups in science, art, and mathematics

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http://egunawan.github.io/algebra

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Applications of groups

We will see applications of group theory in 3 areas:

1. Science
2. Art
3. Mathematics

Our choice of examples is influenced by how well they illustrate the material rather than how useful they are.

Groups of symmetries

Intuitively, something is symmetrical when it looks the same from more than one point of view.

Can you think of an object that exhibits symmetry?

The examples of groups that we've seen so far deal with arrangements of similar things.

We will see the following fact later.

Cayley's Theorem

Every group can be viewed as a collection of ways to rearrange some set of things.

How to make a group out of symmetries

Groups relate to symmetry because an object's symmetries can be described using arrangements of the object's parts.

The following algorithm tells us how to construct a group that describes (or measures) a physical object's symmetry.

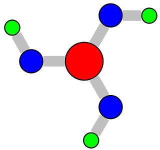
Algorithm

1. Identify all the parts of the object that are similar (e.g., the corners of an n-gon), and give each such part a different number.
2. Consider the actions that may rearrange the numbered parts, but leave the object in the same physical space. (This collection of actions forms a group.)
3. If you want to visualize the group, explore and map it as we did in the previous lecture with the rectangle puzzle, etc.

Shapes of molecules

Because the shape of molecules impacts their behavior, chemists use group theory to classify their shapes.

For example, the following figure depicts a molecule of Boric acid, $B(OH)_3$.



Note that a mirror reflection is *not* a symmetry of this molecule.

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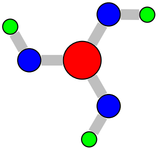
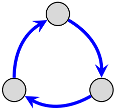


Algorithm

1. Identify all the parts of the object that are similar, and give each such part a different number.
2. Consider the actions that may rearrange the numbered parts, but leave the object in the same physical space.
3. If you want to visualize the group, explore and create a Cayley diagram.

The group of symmetries of Boric acid has 3 actions requiring at least one generator. If we choose "120° clockwise rotation" as our generator, then the actions are:

1. the *identity* (or "do nothing") action: e
2. 120° clockwise rotation: r
3. 240° clockwise rotation: r^2.

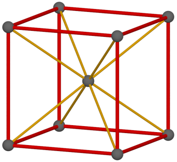


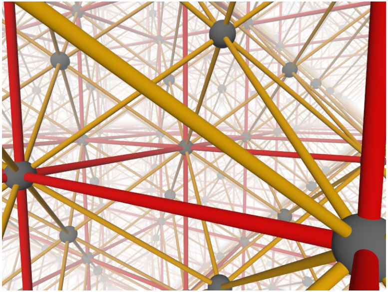
Comments

- Step 1 of our Algorithm numbers the object's parts so that we can track the manipulations permitted in Step 2. Each new state is a rearrangement of the object's similar parts and allows us to distinguish each of these rearrangements; otherwise we could not tell them apart.
- Not *every* rearrangement is valid. We are only allowed actions that maintain the object's physical integrity *and* leave the object taking up the same space as it did originally (that is, preserving its the object's footprint).
- Step 2 requires us to find *all* actions that preserve the object's footprint and physical integrity; not just the generators.
- To complete Step 3 (make a Cayley diagram), we must make a choice concerning generators. Different choices in generators may result in different Cayley diagrams.
- Select as small a generating set as possible. We can never choose too many generators, but we can choose too few. However, having "extra" generators only clutters our Cayley diagram.

Crystallography

Solids whose atoms arrange themselves in a regular, repeating pattern are called crystals. The study of crystals is called crystallography.





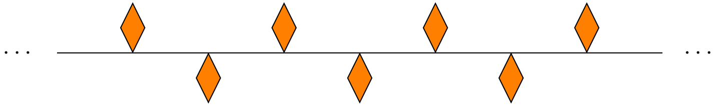
When chemists study such crystals they treat them as patterns that repeat without end. This allows a new manipulation that preserves the infinite footprint of the crystal and its physical integrity: translation.

In this case, the groups describing the symmetry of crystals are infinite. Why?

Frieze patterns

Crystals are patterns that repeat in 3 dimensions. Patterns that only repeat in 1 dimension are called frieze patterns. The groups that describe their symmetries are called frieze groups.

Frieze patterns (or at least finite sections of them) occur throughout art and architecture. Here is an example:

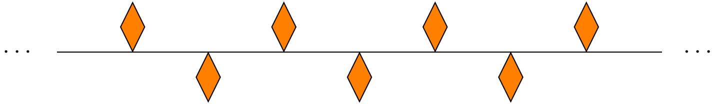


This frieze admits a new type of manipulation that preserves its footprint and physical integrity: a glide reflection, which consists of an appropriate horizontal translation followed by a vertical flip.

Note: a vertical flip all by itself does not preserve the footprint, and thus is not one of the actions of the group of symmetries.

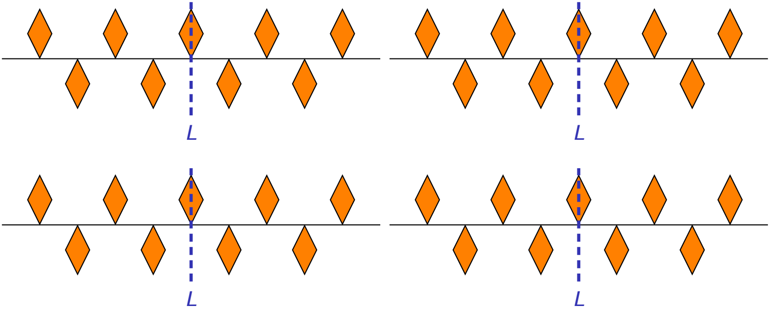
Frieze patterns

Here is an example of a frieze pattern.



Example

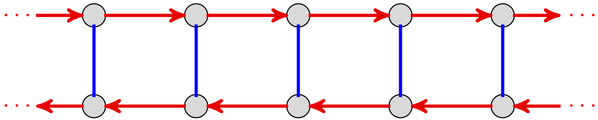
Determine the group of symmetries of this frieze pattern and draw a possible Cayley diagram.



Frieze patterns

The group of symmetries of the frieze pattern on the previous slide turns out to be infinite, but we only needed two generators: horizontal flip and glide reflection.

Here is a possible Cayley diagram:

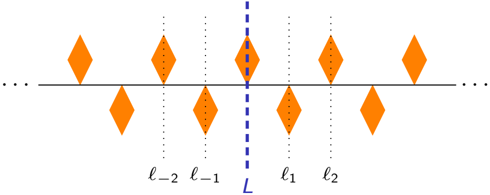


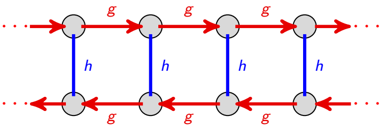
Observations:

- The horizontal flip is its own inverse.
- If you only do glide reflections, you will never get back to where you started.
- The horizontal flip and glide reflection do not commute.

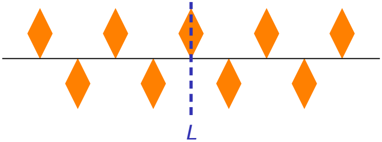
HW hints

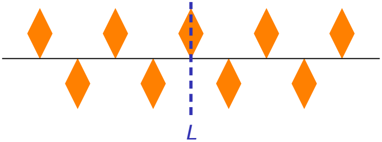
Consider the same frieze pattern and the Cayley diagram of its symmetry group.

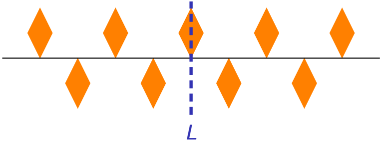


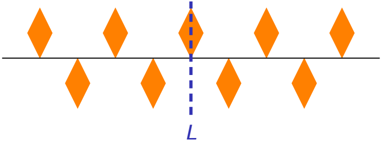


Let g be a glide-reflection to the right, and h a horizontal flip about the thick dashed line. Express the reflection about ℓ_1 and ℓ_2 in terms of g and h.



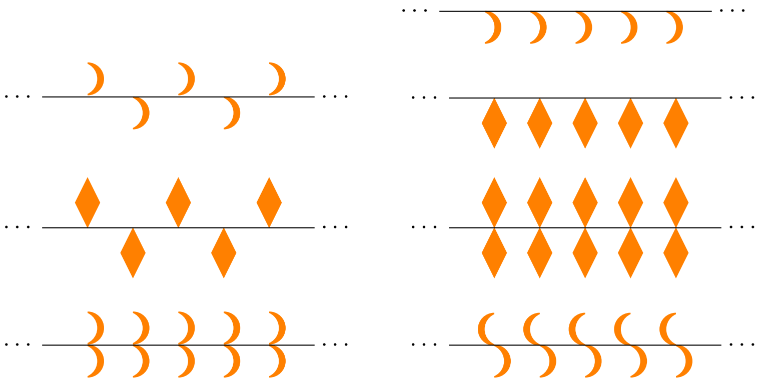




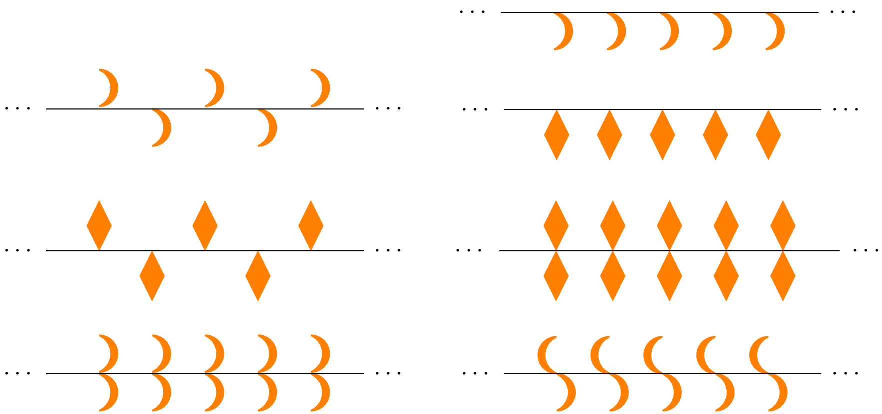


Frieze patterns

- The symmetry of any frieze pattern can be described by one of 7 different infinite groups.
- Warning: Some frieze groups are *isomorphic* (have the same structure) even though the visual appearance of the patterns (and Cayley graphs) may differ.



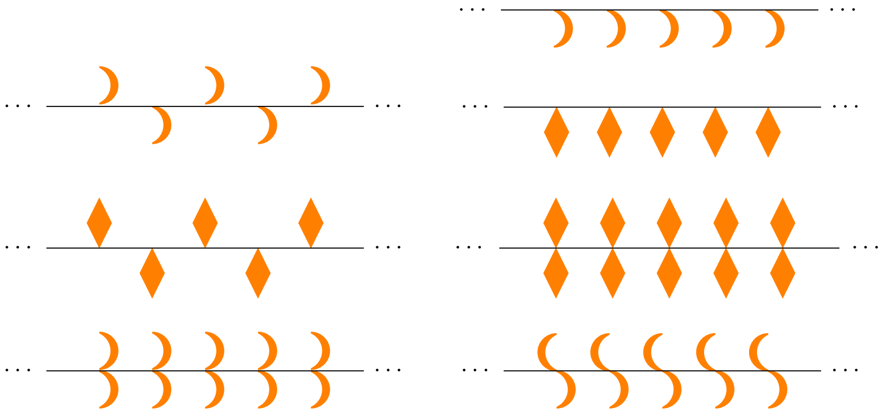
The 7 types of frieze patterns (HW hints)



Questions

- What basic types of symmetries (translation, horizontal reflection, vertical reflection, rotation, glide reflection) do these frieze groups have?

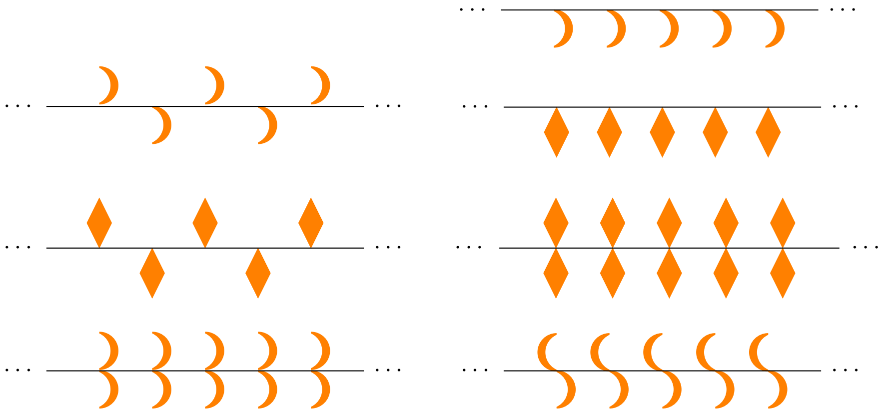
The 7 types of frieze patterns (HW hints)



Questions

- Give a (minimal) generating set for each frieze group
- Draw a Cayley diagram for each frieze group

The 7 types of frieze patterns)

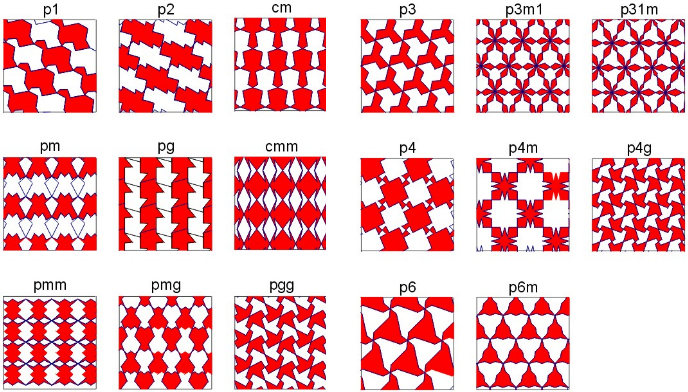


Questions

- Which of these frieze groups are abelian?
- Which of these frieze patterns have isomorphic frieze groups?

The 17 types of wallpaper patterns

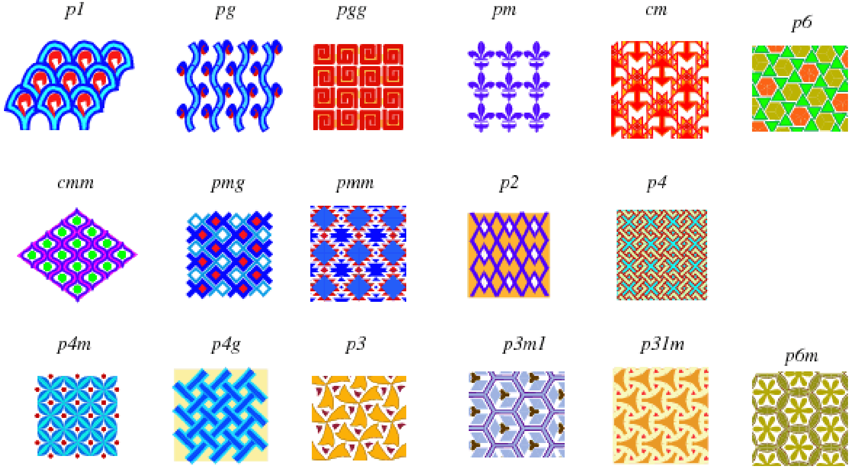
- The symmetry of 2-dimensional repeating patterns, called "wallpaper patterns," has also been classified. There are 17 different wallpaper groups.
- There are 230 crystallographic groups, which describe the symmetries of 3-dimensional repeating patterns.



Images courtesy of Patrick Morandi (New Mexico State University).

The 17 types of wallpaper patterns

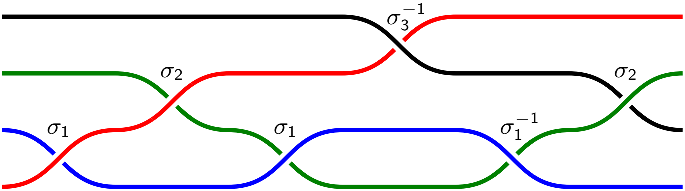
Here is another picture of all 17 wallpapers, with the official IUC notation for the symmetry group, adopted by the International Union of Crystallography in 1952.



Braid groups

Another area where groups arise in both art and mathematics is the study of braids.

The following is a picture of an element (action) from the braid group $B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$:



The braid $b = \sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2 = \sigma_1\sigma_2\sigma_3^{-1}\sigma_2$.

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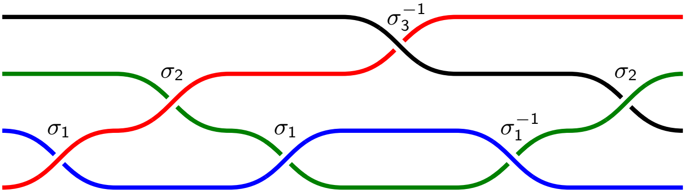


The braid $b = \sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2 = \sigma_1\sigma_2\sigma_3^{-1}\sigma_2$.

Why does the set of braids on n strings forms a group?

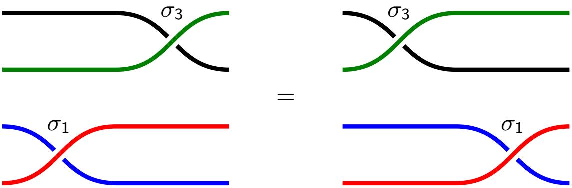
To combine two braids, just concatenate them.

Every braid is reversible – just "undo" each crossing. In the example above,

$$e = bb^{-1} = (\sigma_1\sigma_2\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2)(\sigma_2^{-1}\sigma_1\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}).$$

Braid groups

There are two fundamental relations in braid groups:



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(if $|i - j| \geq 2$)

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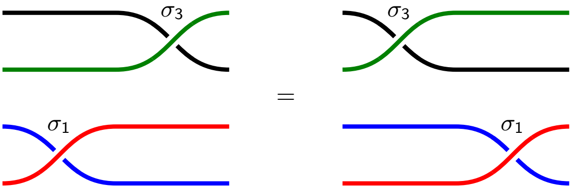

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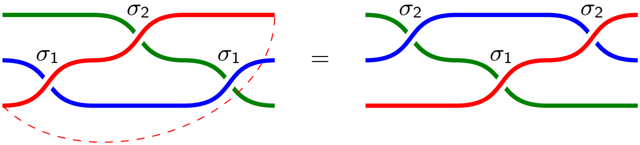

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

We can describe the braid group B_4 by the following presentation:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_3 = \sigma_3\sigma_1, \;\; \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \;\; \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle.$$

We will study presentations more in the next lecture; this is just an introduction.