Math 3230 Abstract Algebra I
Section 1.1 Groups and Cayley graphs

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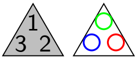
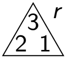
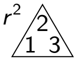
The “Triangle Puzzle” group

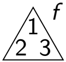
Recall D_3 the collection of symmetries of an equilateral triangle: 

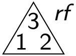
There are 6 rigid motions: clockwise rotations by 0^o, 120^o, 240^o, and three reflections.

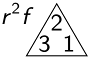
If we let r be the a (clockwise) 120° rotation and f be a horizontal flip with the vertical axis as the mirror, we can get to each of the 6 actions by combining a sequence of f and r (read from left to right):

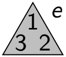
- A (clockwise) 120° rotation: r 

; A (clockwise) 240° rotation: r^2 

- A horizontal flip: f 

; Rotate + horizontal flip: rf 

- Rotate twice + horizontal flip: r^2f 

- The identity action: e 

One set of generators: $D_3 = \langle r, f \rangle$.

Our informal definition of a group

Rule 1
There is a predefined list of actions that never changes.

Rule 2
Every action is reversible.

Rule 3
Every action is deterministic.

Rule 4
Any sequence of consecutive actions is also an action.

Definition (informal)
A group is a set of actions satisfying Rules 1–4.

Examples

1. Place a US quarter and a Canadian quarter side by side on your desk. Consider just one action of swapping the places of the two coins. Is this a group?

2. Imagine you have five quarters in your left pocket. Consider two actions, moving a quarter from your left pocket into your right pocket and moving a quarter from your right pocket into your left pocjet. Is this a group?

Summary of the big ideas

Loosing speaking a group is a **set of actions** satisfying some properties: deterministic, reversibility, and closure.

A generating set for a group is a **subcollection of actions** that together can produce all actions in the group – like a spanning set in a vector space.

Usually, a generating set is *much smaller* than the whole group.

Given a generating set, the individual actions are called generators.

Example

The set of all possible rigid motions with respect to an equilateral triangle is an example of a group. Two actions are the same if they have the *same "net effect"*, e.g., rotating by 120^o once vs. rotating by 120^o 4 times.

Note that the group is the set of actions one can perform, ***not*** the set of configurations of the triangle. However, there is a bijection between these two sets.

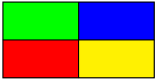
The triangle puzzle group has 6 actions but we can find a generating set of size 2.

The Rectangle Puzzle group

- Consider a clear glass **rectangle** and label it as follows:

1	2
4	3

If you prefer, you can use colors instead of numbers:



We'll use numbers, and call the above configuration the **solved state** of our puzzle.

- The idea of the game is to scramble the puzzle and then find a way to return the rectangle to its original solved state.

- Our "predefined list" consists of two actions: horizontal flip and vertical flip.

  Loosely speaking, we allow these moves because they are rigid motion (i.e. they preserve the symmetry of the rectangle). Do you see other rigid motions?

Road map for The Rectangle Puzzle

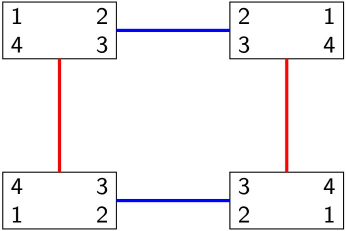
For covenience, let's say that when we flip the rectangle, the numbers automatically become "right-side-up," as they would if you rotate a smart phone.

Check that, using only sequences of horizontal and vertical flips, we can obtain only four configurations.

The following is a "road map" of the rectangle puzzle.

e: identity

h: horizontal flip



v: vertical flip

r: 180° rotation

Observations? What sorts of things does the map tell us about the group?

Observations

Let G denote the rectangle group. This is a **set** of four actions. We see:

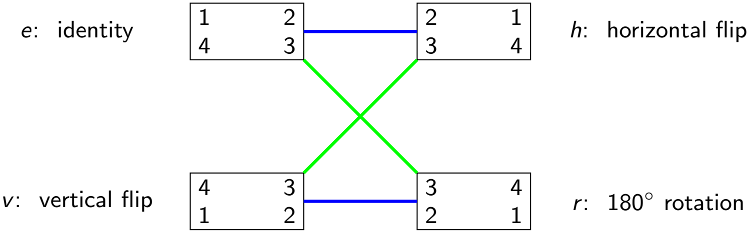
- G has 4 actions: the "*identity*" action e, a horizontal flip h, a vertical flip v, and a 180° rotation r.
$$G = \{e, h, v, r\}\,.$$
- We need two actions to "generate" G. In our diagram, each **generator** is represented by a different type (color) of arrow. We write:
$$G = \langle h, v \rangle\,.$$
- The map shows us how to get from any one configuration to any other. There is more than one way to follow the arrows! For example
$$r = hv = vh\,.$$
- For this particular group, the order of the actions is irrelevant! We call such a group **abelian**. Note that the triangle puzzle group D_3 is *not* abelian.
- Every action in G is its own **inverse**: That is,
$$e = e^2 = h^2 = v^2 = r^2\,.$$
The triangle puzzle group D_3 does **not** have this property. Algebraically, we write:
$$e^{-1} = e, \qquad v^{-1} = v, \qquad h^{-1} = h, \qquad r^{-1} = r\,.$$

An alternative set of generators for the Retangle Puzzle

The rectangle puzzle can also be generated by a horizontal flip and a 180° rotation:

$$G = \langle h, r \rangle .$$

Let's build a Cayley graph using this alternative set of generators.



Do you see this road map has the "same structure" as our first one? Of course, we need to "untangle it" first.

Perhaps surprisingly, this might *not* always be the case.

That is, there are (more complicated) groups for which different generating sets yield road maps that are structurally different. We'll see examples of this shortly.

Cayley diagrams

As we saw in the previous example, how we choose to layout our map is irrelevant.

What is important is that the connections between the various states are preserved.

However, we will attempt to construct our maps in a pleasing to the eye and symmetrical way.

The official name of the type of group road map that we have just created is a Cayley diagram, named after 19th century British mathematician Arthur Cayley.

In general, a Cayley diagram consists of nodes that are connected by colored (or labeled) arrows, where:

- an arrow of a particular color represents a specific generator;
- each action of the group is represented by a unique node (sometimes we will label nodes by the corresponding action).
- Equivalently, each action is represented by a (non-unique) path starting from the **solved state** (or original configuration).

Hints for HW 1

Let r be a clockwise rotation by $2\pi/6$ radians (60^o) of a regular 6-gon. This generates a group denoted by $C_6 = \langle r \rangle$ which consists of the 6 rotating actions $\{e, r, r^2, r^3, r^4, r^5\}$.

1. Draw the original configuration of the hexagon and also the other 5 configurations that you would get after applying the 5 non-identity rotations.

2. Draw a Cayley diagram for C_6 with $\{r\}$ as the generating set.

3. Is the group $\langle r^5 \rangle$ (generated by a 300^o rotation) the same as C_6?

4. Prove the following.
   - The group $\langle r^2 \rangle$ (generated by a 120^o rotation) has three elements.
   - The group $\langle r^3 \rangle$ (generated by a 180^o rotation) has two elements.

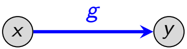
   Hence $\{r^2\}$ and $\{r^3\}$ are not generating sets of C_6.
5. How many elements does the group $\langle r^3, r^4 \rangle$ have?

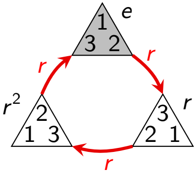
Cayley diagrams (Review)

In general, a Cayley diagram consists of nodes that are connected by colored (or labeled) arrows, where:

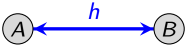
- an arrow of a particular color represents a specific generator;
- each action of the group is represented by a unique node (sometimes we will label nodes by the corresponding action).
- Equivalently, each action is represented by a (non-unique) path starting from the **solved state** (or original configuration).

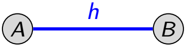
More on arrows

- An arrow corresponding to the generator g from node x to node y means that node y is the result of applying the action $g \in G$ to node x: 

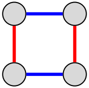
  Example: 

- If an action $h \in G$ is its own inverse (that is, $h^2 = e$), then we have a 2-way arrow. Our convention is to drop the tips on all 2-way arrows. Thus, these are exactly the same:





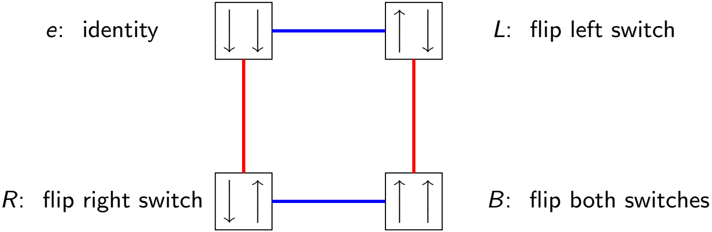
When we want to focus on a group's structure, we frequently omit the labels at the nodes. Thus, the Cayley diagram of the rectangle puzzle can be drawn as follows:



The 2-Light Switch group

Let's map out another group, which we'll call the *2-Light Switch Group*:

- Consider two light switches side by side that both start in the off position (This is our "solved state" or "original configuration").
- We are allowed 2 actions: flip L switch and flip R switch.



Notice how the Cayley diagrams for the Rectangle Puzzle $G = \{e, v, h, r\}$ and the 2-Light Switch Group $G' = \{e, L, R, B\}$ are essentially the same.

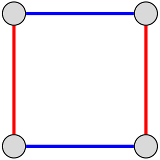
Although these groups are superficially different, the Cayley diagrams help us see that *they have the same structure*. (The fancy phrase for this property is that the "*two groups are isomorphic*"; more on this later.)

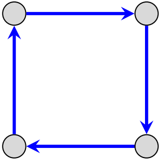
The Klein 4-group

Any group with the same Cayley diagram as the Rectangle Puzzle and the 2-Light Switch Group is called the Klein 4-group, denoted by V_4 for *vierergruppe*, "four-group" in German. It is named after the mathematician Felix Klein.

It is important to point out that the number of different types (i.e., colors) of arrows matters. For example, the Cayley diagram on the right *does not* represent V_4.

V_4





What is G?

Question 1

- What group has a Cayley graph like the diagram on the right?

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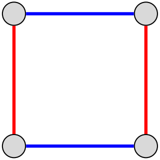
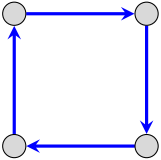
V_4





What is G?

Question 2

- How would you give a proof (that is, a convincing argument) that these two groups have truly different structures?
  Hint: Can you find a property that one group has that the other does not?

The Klein 4-group

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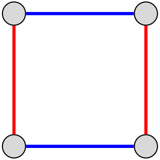
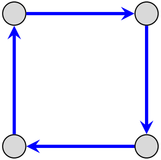
V_4





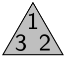
What is G?

Question 3

- Can you find another group of size 4 that is different from both of these? (More on this later in HW 2)

The "Triangle Puzzle" group

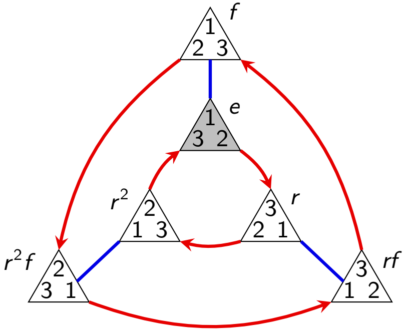
Back to D_3, the symmetries of an equilateral triangle:



Recall that D_3 has 6 actions:

- The identity action: e
- A (clockwise) 120° rotation: r
- A (clockwise) 240° rotation: r^2
- A horizontal flip: f
- Rotate + horizontal flip: rf
- Rotate twice + horizontal flip: r^2f.

One set of generators: $D_3 = \langle r, f \rangle$.



Notice that multiple paths can lead us to the same node. These give us **relations** in our group. For example:

$$r^3 = e, \qquad r^{-1} = r^2, \qquad f^{-1} = f, \qquad rf = fr^2, \qquad r^2f = fr.$$

This group is **non-abelian**: $rf \neq fr$.

The "Triangle Puzzle" group (Hints for HW 1)

Let h be the reflection of the triangle that fixes the lower-right corner, that is, $h = rf$.

1. Write all 6 actions of D_3 using only f and h (as opposed to the previous slides, where all actions are written using only f and r).

2. Draw a Cayley diagram using f and h as generators.

3. We see that $\{r, f\}$ and $\{f, h\}$ are minimal generating sets of D_3. Find other minimal generating sets of D_3.

Properties of Cayley graphs

Observe that at every node of a Cayley graph, there is exactly one out-going edge of each color.

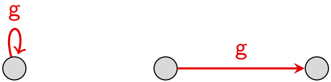
Question 1

Can an edge in a Cayley graph ever connect a node to itself?

Question 2

Suppose we have an edge corresponding to generator g that connects a node x to itself. Does that mean that the edge g connects *every* node to itself? In other words, can an action be the *identity action* when applied to some actions (or configurations) but not to others?

Visually, we're asking if the following scenerio can ever occur in a Cayley diagram:



A Theorem and Proof!

Perhaps surprisingly, the previous situation is *impossible*! Let's properly formulate and prove this.

Theorem

Suppose an action g has the property that $gx = x$ for some action x. Then g is the *identity action*, i.e., $gh = h = hg$ for all actions h.

Proof

The identity action (we'll denote by 1) is simply the action hh^{-1}, for any action h.

By assumption, we have

$$gx = x.$$

By Rule 4 (of a group), the inverse of x, denoted by x^{-1}, exists. Multipling by x^{-1} *on the right* yields:

$$g = gxx^{-1} = xx^{-1} = 1.$$

Thus g is the identity action. $\square$

This was our first mathematical proof! It shows how we can deduce interesting properties about groups which were not explicitly *built into* the rules *from* the rules.