# Sec 4.2 Kernels

**Prop 2** A gp hom $f: G_1 \to G_2$ is injective iff $\ker f = \{e_1\}$

"Kernel of $f$ is the trivial subgroup"

**Pf** ($\Rightarrow$) Suppose $f$ is injective. (i.e. show $f(a)=f(b) \Rightarrow a=b$)

exercise

Let $a \in \ker f := \{ b \in G_1 : f(b) = e_2 \}$

Then $f(a) = e_2$ by def of ker $f$

$= f(e_1)$ by Prop 1 part 1 (previous section)

Since $f$ is injective, $a = e_1$.

So $\ker f \subseteq \{e_1\}$.

But $\ker f$ is a group by Prop 1 part 4,

So $e_1 \in \ker f$, so $\ker f = \{e_1\}$.

($\Leftarrow$) Suppose $\ker f = \{e_1\}$.

Suppose $a, b \in G_2$ s.t $f(a) \overset{(*)}{=} f(b)$.

Then

$$
\begin{aligned}
f(ab^{-1}) &= f(a) f(b^{-1}) && \text{since } f \text{ is a gp hom} \\
&= f(a)[f(b)]^{-1} && \text{by Prop 1 part 2} \\
&= f(a)[f(a)]^{-1} && \text{by } (*) \\
&= e_2.
\end{aligned}
$$

Thus $ab^{-1} \in \ker(f) = \{e_1\}$ by assumption

Then $ab^{-1} = e_1$.

Hence $a = b$.

So $f$ is injective. $\square$

**E.g 1**

Determine all possible homs $f: \mathbb{Z}_7 \to \mathbb{Z}_{12}$.

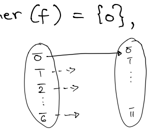
- By Prop 1 part 4, the kernel of $f$ must be a subgroup of $\mathbb{Z}_7$.

  There are only two subgroups of $\mathbb{Z}_7$,

  $\{0\}$ and itself because $\langle g \rangle = \mathbb{Z}_7$ for all $g \in \{\bar{1}, \bar{2}, \bar{3}, \ldots, \bar{6}\}$

- By Prop 1 part 3, the image of a subgroup of $\mathbb{Z}_7$ must be a subgroup of $\mathbb{Z}_{12}$.

  If $\ker(f) = \{0\}$, then $f$ is injective by prev prop,



  So $f(\mathbb{Z}_7)$ has order 7

  (each elt of $\mathbb{Z}_7$ is sent to a unique elt in $\mathbb{Z}_{12}$).

  But no subgroup of $\mathbb{Z}_{12}$ has order 7 by the Lagrange Thm.

  So $\ker(f) = \mathbb{Z}_7$.

  The only possible hom $\mathbb{Z}_7 \to \mathbb{Z}_{12}$ is the zero map, $f(a) = \bar{0}\ \forall a \in \mathbb{Z}_7$.

ended here wk 10 Wed —

## E.g 2

$f: GL_2(\mathbb{R}) \longrightarrow \mathbb{R}^*$ defined by $f: A \mapsto \det(A)$

$GL_2(\mathbb{R})$: 2x2 matrix with det non zero w/ product as binary operation, $Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbb{R}^*$: non zero real #s w/ mult as binary operation, $Id = 1$

Note $f$ is a gp hom because

$\boxed{HW}$ $\det(AB) = \det(A)\det(B)$

$\ker f = SL_2(\mathbb{R})$ group of 2x2 matrices w/ det 1.

$D_1 = \det \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = a_1 d_1 - b_1 c_1$

$D_2 = \det \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = a_2 d_2 - b_2 c_2$

$D_1 D_2 = a_1 d_1 a_2 d_2 - a_1 d_1 b_2 c_2 - a_2 d_2 b_1 c_1 + b_1 b_2 c_1 c_2$

$$AB = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}$$

$$\det(AB) = \cancel{a_1 a_2 c_1 b_2} + \boxed{a_1 a_2 d_1 d_2} + \boxed{b_1 c_2 c_1 b_2} + \cancel{b_1 c_2 d_1 d_2} - \left(\cancel{c_1 a_2 a_1 b_2} + c_1 a_2 b_1 d_2 + d_1 c_2 a_1 b_2 + \cancel{d_1 c_2 b_1 d_2}\right)$$

$$= \det(A)\det(B)$$

## E.g 3

$G$ group & $g \in G$.

Let $f: \mathbb{Z} \longrightarrow G$ group hom defined by $f(n) = g^n$

Case 1 order of $g$ is infinite.

That is, $g^k \neq e \quad \forall \ k \in \mathbb{Z}$.

Since $f(\mathbb{Z}) = \{\ldots, g^{-2}, g^{-1}, e, g, g^2, \ldots\} = \langle g \rangle$ cyclic subgroup generated by $g$,

$\ker(f) = \{0\}$

Case 2 order of $g$ is finite, say $k$.

Then $g^k = e$ and if $0 < k' < k$ then $g^{k'} \neq e$

Then $\ker(f) = \{$ all integer multiples of $k\}$

$= k\mathbb{Z}$.

**Prop 3** If $f: G_1 \to G_2$ is a group homomorphism, then $\ker(f) \trianglelefteq G_1$.

**Pf** We will show that $gkg^{-1} \in \ker(f)\ \forall\ g \in G_1,\ k \in \ker(f)$.

Let $k \in \ker(f)$, $g \in G_1$.

Then $f(gkg^{-1}) = f(g)f(k)f(g^{-1})$ since $f$ is gp hom

$= f(g)\,f(k)\,[f(g)]^{-1}$ by Prop 1

$= f(g)\,e_1\,[f(g)]^{-1}$ since $k \in \ker(f)$

$= e_2$

$\therefore\ gkg^{-1} \in \ker(f)$ $\square$

BIG OBSERVATION: Given any gp hom $f: G_1 \to G_2$, we can always form the quotient group $G_1/\ker(f)$

Recall: $G/H = \{\text{left cosets of } H\} = \{xH \mid x \in G\}$

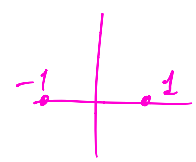
$G/H$ is a group iff $H$ is normal

**E.g. 4**

Define $\phi: D_4 \longrightarrow C_2$

$D_4 = \langle r, f \rangle = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$

$C_2 = \{e^{0\pi i}, e^{1\pi i}\} = \{1, -1\}$




by $\phi(r) = 1$

$\phi(f) = -1$

Then all rotations are sent to 1, since $\phi(r^k) = [\phi(r)]^k = 1^k = 1$

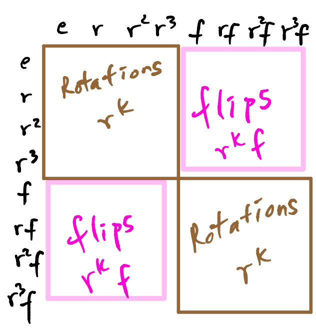
all reflections are sent to $-1$, since $\phi(r^kf) = \phi(r^k)\phi(f)$

$= 1 \cdot -1$

$= -1$

So $\ker(\phi) = \{\text{all rotations}\}$

Visualize the quotient $D_4/\ker(\phi)$ using the multp table:




Compare w/ the table for $C_2$

| | 1 | -1 |
|---|---|---|
| 1 | 1 | -1 |
| -1 | -1 | 1 |

Optional (for FTH)

$ij = k,\ jk = i,\ ki = j$

$ji = -k,\ kj = -i,\ ik = -j$

**E.g. 5** Define $f : Q_8 \longrightarrow V_4 = \{e, v, h, r\}$ by

vertical flip, horizontal flip, rotation

$$i \longmapsto v$$
$$j \longmapsto h$$

$f(1) = e$

$f(-1) = f(i^2) = f(i)f(i) = v^2 = e$

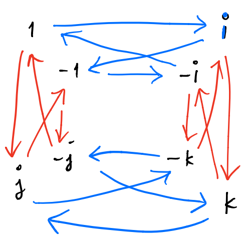
$f(k) = f(ij) = f(i)f(j) = vh = r$

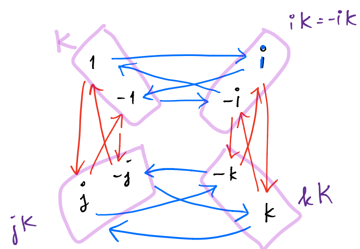
$f(-k) = f(-1)\,f(k) = er = r$

$f(-i) = f(-1)\,f(i) = ev = v$

$f(-j) = f(-1)\,f(j) = eh = h$

$\mathrm{Ker}(f) = \{1, -1\} = K$




Use the "visual" quotient concept to collapse cosets into single nodes

$$\begin{array}{ccc} K & = & iK \\ \| & & \| \\ jK & = & kK \end{array}$$

to get the Cayley diagram for $Q_8/K$

Compare w/ the Cayley diagram for $V_4$

$$\begin{array}{ccc} e & = & v \\ \| & & \| \\ h & = & r \end{array}$$

**Note:** $\mathrm{Im}(f) = V_4$ and

$Q_8/\mathrm{Ker}(f)$ looks just like $V_4$

In general, $\dfrac{\text{Domain}}{\mathrm{Ker}(f)} \cong \mathrm{Im}(f)$

by $\qquad x\,\mathrm{Ker}(f) \longmapsto x$