Sec 3.7 Conjugacy classes (con't w/ the theme of normal subgroup & cosets

Recall (Sec 3.3 "Normal Subgroups")

- For a fixed $g \in G$, the set $gHg^{-1} \stackrel{def}{=} \{ghg^{-1} \mid h \in H\}$ is called the <u>conjugate</u> <u>of H</u> <u>by g</u> and it's a subgroup.
- H is <u>normal</u> iff $gHg^{-1} = H$ for all $g \in G$.

Today, instead of fixing $H \underset{\text{subgroup}}{<} G$, we can fix $x \in G$ and define $cl_G(x) \stackrel{def}{:=} \{gxg^{-1} \mid g \in G\}$, the conjugacy class of x

$= \{y \in G \mid \exists g \in G \text{ with } y = gxg^{-1}\}$

I.e. an elt of G lives in cl_G iff it can be written as gxg^{-1} for some $g \in G$.

<u>Ex</u> • $cl_G(e) = \{\underbrace{geg^{-1}}_{e} \mid g \in G\} = \{e\}$

- If $x \in G$ commutes w/ every elt in G, then $gxg^{-1} = gg^{-1}x = x$ so $cl_G(x) = \{x\}$

<u>Def</u> We say $x, y \in G$ are <u>conjugate</u> ("x is a conjugate of y") if $\exists\ g \in G$ s.t $gxg^{-1} = y$

<u>Prop</u> 1 The above relation in an equivalence relation

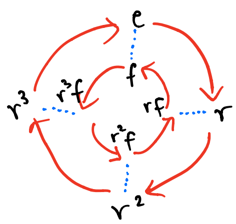
<u>Pf.</u> • Reflexive : $x = exe^{-1}$ so x is a conjugate of itself

- Symmetric : If x is a conjugate of y ($x = gyg^{-1}$ for some $g \in G$) then $xg = gy$, so $g^{-1}xg = y$ (meaning y is a conjugate of x)
- Transitive : If $x = gyg^{-1}$ for some $g \in G$ and $y = hzh^{-1}$ for some $h \in G$ then $x = ghzh^{-1}g^{-1}$ $= (gh)z(gh)^{-1}$

exercise

$\therefore$ This relation partitions G into equivalence classes (conjugacy classes).

Rem If G is abelian, every conjugacy class is of the form $Cl_G(x)=\{x\}$.



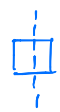
Ex Compute the conjugacy classes of $D_4=\{e,r,r^2,r^3,\ f,rf,r^2f,r^3f\}$

- $Cl_{D_4}(r)=\{r,\ r^3\}$

Computation: only need to check grg^{-1} for g that doesn't commute w/ r

$frf^{-1}=r^3$, $(rf)r(rf)^{-1}=r^3$, $(r^2f)r(r^2f)^{-1}=r^3$, $(r^3f)r(r^3f)^{-1}=r^3$.

- $Cl_{D_4}(f)=\{f,\ r^2f\}$  f

The elts that commute with f are  $=r^2f$, $180°$ rot $=r$, e,f

(don't need to check gfg^{-1} for those g)

Check that $rfr^{-1}=r^2f$

$r^3f(r^3)^{-1}=r^2f$

$(rf)f(rf)^{-1}=r^2f$

$(r^3f)f(r^3f)^{-1}=r^2f$

- $cl_{D_4}(rf)=\{rf,\ r^3f\}$

Partition of D_4 by its conjugacy classes

e	r	f r^2f
r^2	r^3	rf r^3f

Prop 2 Every normal subgroup is the union of a collection of conjugacy classes.

i.e., if $N \triangleleft G$ and $x\in N$, then $cl_G(x)\subset N$.

Pf Suppose $N\triangleleft G$. Let $x\in N$. Then $gxg^{-1}\in N$ for all $g\in G$ (since $N\triangleleft G$).

Thus, $cl_G(x):=\{gxg^{-1}\mid g\in G\}\subset N$.

—— ended here week 9 Wed ——

Prop 3 Conjugate elements have the same order, i.e.
If $|x| = n$ then $|gxg^{-1}| = n$ for all $g \in G$

Pf Let $g \in G$.
Let $|x| = n$. This means n is the smallest pos integer s.t $x^n = e$.
Then $(gxg^{-1})^n = (gxg^{-1})(gxg^{-1})\ldots(gxg^{-1}) = gx^ng^{-1} = geg^{-1} = e.$
This means $|x| \geq |gxg^{-1}|$.

If $|gxg^{-1}| = m$, then $x^m = gx^mg^{-1} = (gxg^{-1})(gxg^{-1})\ldots(gxg^{-1}) = e.$
Therefore $|x| \leq |gxg^{-1}|$.

Def For $\sigma \in S_n$, the cycle type of σ is a list $c_1, c_2, \ldots, c_n$, where c_i is the number of cycles of length i in σ

E.g

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
• $(18)(5)(23)(4967)$ has cycle type	1	2	0	1	0	0	0	0	0
• $(1\,8\,4\,2\,3\,4\,9\,6\,7)$ has cycle type	0	0	0	0	0	0	0	0	1
• $e = (1)(2)(3)(4)(5)(6)(7)(8)(9)$ — " —	9	0	0	0	0	0	0	0	0

Lemma 4 (HW 3) For any $\sigma \in S_n$,
$$\sigma^{-1}(a_1\, a_2 \ldots a_k)\,\sigma = (\sigma(a_1)\,\sigma(a_2)\ldots\sigma(k))$$

This means that every k-cycle is conjugate to any other k-cycle

E.g. $x = (12)$, $y = (14) \in S_6$ are both 2-cycles, so they must be conjugate:
Find $\sigma \in S_6$ s.t $\sigma^{-1}(12)\sigma = (14)$

We need $a_1 = 1$, $a_2 = 2$ and $\begin{pmatrix}\sigma(1) = 1 \\ \sigma(2) = 4\end{pmatrix}$ OR $\boxed{\begin{matrix}\sigma(1) = 4 \\ \sigma(2) = 1\end{matrix}}$

Let $\sigma(1) = 4$
$\sigma(2) = 1$
$\boxed{\begin{matrix}\sigma(3) = 2 \\ \sigma(4) = 3 \\ \sigma(5) = 5 \\ \sigma(6) = 6\end{matrix}}$ doesn't matter as long as $\sigma \in S_n$

So σ^{-1}: $4 \mapsto 1$
$1 \mapsto 2$
$2 \mapsto 3$
$3 \mapsto 4$
$5 \mapsto 5$
$6 \mapsto 6$

$\sigma = (4\,3\,2\,1)(5)(6)$ $\sigma^{-1} = (1\,2\,3\,4)(5)(6)$

Check:
$\sigma^{-1}(12)\sigma =$
$(1234)(12)(4321) = (14)$

:)

<u>Thm 5</u> $x, y \in S_n$ are conjugate iff they have the same cycle type

<u>Proof</u>

$(\Rightarrow)$

Suppose $x = (a_1\, a_2 \ldots a_k)(b_1\, b_2 \ldots b_t)\ldots$

and suppose $\sigma^{-1} x \sigma = y$.

Then by Lemma 4, $y = (\sigma(a_1)\, \sigma(a_2) \ldots \sigma(a_k))(\sigma(b_1)\, \sigma(b_2) \ldots \sigma(b_t)) \ldots$

so y has the same cycle type as x.

$(\Leftarrow)$

Suppose $x = (a_1\, a_2 \ldots a_k)(b_1\, b_2 \ldots b_t)\ldots$

and $y = (a_1'\, a_2' \ldots a_k')(b_1'\, b_2' \ldots b_t')$

have the same cycle type.

To show that they are conjugate, let

$$\sigma : \begin{array}{l} a_1 \mapsto a_1' \\ a_2 \mapsto a_2' \\ \vdots \\ a_k \mapsto a_k' \\ b_1 \mapsto b_1' \\ b_2 \mapsto b_2' \\ \vdots \\ b_t \mapsto b_t' \\ \vdots \end{array} \quad \text{then} \quad \sigma^{-1} : \begin{array}{l} a_1' \mapsto a_1 \\ a_2' \mapsto a_2 \\ \vdots \\ a_k' \mapsto a_k \\ b_1' \mapsto b_1 \\ \vdots \end{array}$$

Then $\sigma^{-1} x \sigma : \ a_1' \; a_2' \; a_3' \ldots a_k' \quad b_1' \; b_2' \ldots b_t' \quad \ldots$

$= y$

$\square$

<u>Summary</u> ① We can conjugate elts (this sec 3.7)

or ② — " — subgroups (Sec 3.3 "Normal subgroups")

① Conjugating elts defines an equivalence class on G.

- The equivalence classes are called <u>conjugacy classes</u>, & they look like $cl_G(x) = \{gxg^{-1} \mid g \in G\}$
- Conjugate elts have the same order & the "same structure"
- An elt $z \in G$ has a unique conjugate iff $z \in Z(G)$, i.e. z commutes w/ every $g \in G$

② Conjugating subgroups an equivalence class on the set of subgroups of G.

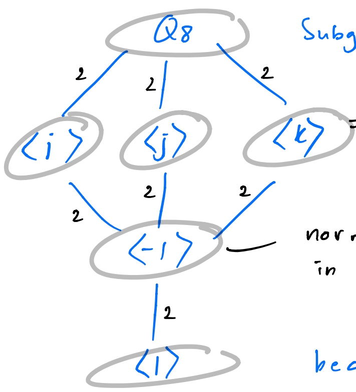
- The equivalence classes have no name/notation, & they look like $\{\underbrace{xHx^{-1}}_{\text{this is called a conjugate subgroup to } H} \mid x \in G\}$
- Conjugate subgroups are isomorphic
- A subgroup $N < G$ has a unique conjugate iff $N \lhd G$.

"will learn more about this later"

E.g. $Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$

Partition the set of subgroups into conjugacy equivalence classes

Subgroup lattice



$= \{k, -1, -k, 1\}$

normal because $-1, 1$ commute w/ all elts in fact, $Z(Q_8) = \{1, -1\}$

because $gNg^{-1} = N \ \forall g \in G$ if $N \lhd G$.