Sec 3.6 Normalizers (and: another application of cosets!)

Idea: • If $H < G$ but $H \ntriangleleft G$ (not normal in G),

we want to measure how far H is from being normal.

• Recall the def: $H \triangleleft G \overset{(def)}{\text{iff}} xH = Hx \quad \forall\, x \in G$.

• One way to measure how far H is from being normal is to check how may elt $x \in G$ satisfy $xH = Hx$

— Think of each elt $x \in G$ as voting yes or no to the normality of H

	vote of x
if $xH = Hx$	YES
if $xH \neq Hx$	NO

• Rem • Every $x \in H$ votes YES. Why? $x \in H \Rightarrow xH = H$ (Sec 3.3)

• Every $x \in G$ votes YES iff $H \triangleleft G$

• If H is not normal, there is at least one elt voting no.

Def (for the elts $x \in G$ which vote Yes in favor of H's normality)

The normalizer of H in G, denoted $N_G(H)$, is

the set $\{ x \in G \mid xH = Hx \}$

$= \{ x \in G \mid xHx^{-1} = H \}$

Slides 3.3 "Normal subgroups"

Wording We say $N_G(H)$ is the set of elements that normalize H.

Prop 1 If $x \in N_G(H)$, then $xH \subseteq N_G(H)$.

Lemma (Prop 2 from Slides 3.2 "Cosets")

• $xH = yH$ for all $y \in xH$ (So it doesn't matter which coset representative you choose, x or y)

• $Hx = Hy$ for all $y \in Hx$ for the same reason

$\boxed{\text{Pf of Prop 1}}$ Suppose $x \in N_G(H)$. Then $xH \overset{(*)}{=} Hx$ by def of $N_G(H)$.

Let $y \in xH$. (Think to self: my goal is to show $y \in N_G(H)$, ie I want to show $yH = Hy$)

Then $yH = xH$ by above lemma

$= Hx$ by $(*)$

$= Hy$ by above lemma $\square$

Rem Prop 1 tells us that members of a left coset vote together as a block: members of xH all vote yes (when $xH = Hx$)

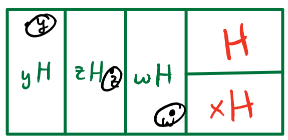
or all vote no (when $xH \neq Hx$)

Prop 2 $|N_G(H)|$ is a multiple of $|H|$.

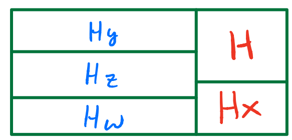
Pf Prop 1 tells us that $N_G(H)$ consists of entire left cosets of H (at least one, H itself).
From Slides 3.2 "Cosets" we know the left cosets are the same size and disjoint.

Cartoon example : Partitions of G

by the left cosets



by the right cosets



The elts of the cosets H and $xH = Hx$ all vote Yes.

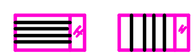
The elts of the left coset yH all vote NO since $yH \neq Hy$.
zH — " —
wH — " —

The normalizer of H in G is $N_G(H) = H \cup xH$

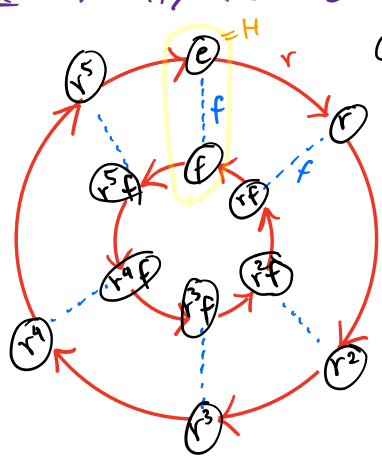
Rem The two extreme cases for $N_G(H)$ are:

- $N_G(H) = G$ iff $H \trianglelefteq G$
- $N_G(H) = H$ (when H is as far from normal as possible)

Cartoon ex.



Ex $H := \langle f \rangle < G := D_6 = \langle f, r \rangle$



Obs The coset r^3H (& H itself) is the only left coset of H which cannot be reached from H by more than one elt of D_6.

E.g. $rH = \{r, rf\}$ can be reached from H by r and by r^5

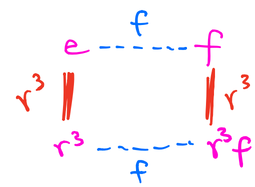
$r^2H = \{r^2f, r^2\}$ can be reached from H by r^2 and r^4

You can check $r^3H = \{r^3f, r^3\} \subset N_G(H)$:

$r^3f\,H = \{r^3f, r^3ff\} = \{r^3f, fr^3f\} = Hr^3f$

and the other 4 left cosets of H are not subsets of $N_G(H)$.
(fyi: $\langle f \rangle$ is not normal in D_6)

So $N_{D_6}(\langle f \rangle) = \langle f \rangle \cup r^3\langle f \rangle$
$= \{e, f, r^3, r^3 f\}$
$\cong V_4$



Exercise

Find a pattern for $N_{D_n}(\langle f \rangle)$ "it depends on whether n is even / odd"

Thm

Let $H < G$. Then $N_G(H) \overset{\text{subgroup of}}{<} G$.

Pf (We need to check all 3 properties of being a subgroup)

Recall $N_G(H) := \{x \in G \mid xHx^{-1} = H\}$.

- **Contains e** $eHe^{-1} = \{ehe^{-1} \mid h \in H\}$
$= H$

- **Inverses exist** Let $x \in N_G(H)$. (We need to show $x^{-1} \in N_G(H)$ that is, show $x^{-1}H(x^{-1})^{-1} = H$)
Then $xHx^{-1} \overset{(*)}{=} H$
So $x^{-1}H(x^{-1})^{-1} = x^{-1}Hx$ (replace $(x^{-1})^{-1}$ with x)
$= x^{-1}(xHx^{-1})x$ (by (*))
$= eHe$
$= H$

- **Closed under the binary operation of G** Suppose $x, y \in N_G(H)$, meaning
$xHx^{-1} \overset{(**)}{=} H$ and $yHy^{-1} \overset{(*)}{=} H$.
We need to show $xy \in N_G(H)$, meaning $xyH(xy)^{-1} = H$.
But $xyH(xy)^{-1} = xyHy^{-1}x^{-1}$ ("shoes-socks property" of inverses: $(xy)^{-1} = y^{-1}x^{-1}$)
$= xHx^{-1}$ (by (*))
$= H$ □ (by (**))

$\boxed{\text{Thm}}$

Cor

Every subgroup of G is normal in its normalizer in G:

If $H < G$, then $\underbrace{H \lhd N_G(H)}_{H \text{ is a normal subgroup of } N_G(H)} < G$

Pf

Let $H < N_G(H)$. To show $H \lhd N_G(H)$, we need to show $xH = Hx$ $\forall x \in N_G(H)$.
But, $xH = Hx$ $\forall x \in N_G(H)$ by def.